DJM3A - OPTIMIZATION TECHNIQUES

Unit I

Linear Programming Problem(LPP): Mathematical Formulation - Graphical Method of Solution – Simplex Method – Big ‘M’ Method – Two Phase Simplex Method – Duality – Dual Simplex Method.

Unit II

Transportation Problems: Mathematical Formulation – Balanced and unbalanced TP – North-West Corner Rule – Least Cost Method – Vogel’s Approximation Method – Test for Optimality – Maximization problems in TP – Assignment Problems(AP): Mathematical Formulation – Method of Solution – Maximization in AP.

Unit III

Sequencing Problems: Introduction – Basic Assumptions – Sequencing n Jobs on 2 Machines – Sequencing n Jobs on 3 machines – Sequencing 2 Jobs on n Machines. Queuing Theory : General Concepts and Definitions – Classification of queues – Poisson Process, Properties of Poisson Process – Queuing Models: 1. (M/M/1):(∞/FCFS), 2. (M/M/1):(N/FCFS), 3. (M/M/c):(∞/FCFS).

Unit IV

Inventory Control: Basics – Types of Inventory Models: Deterministic Models: Model I Purchase Model without Shortages – Model II Production Model without Shortages – Model III Purchase Model with Shortages – Model IV Production Model with Shortages – Model V Price Break Model. Probabilistic Models: Newspaper boy problem – Discrete and Continous type cases – ABC Analysis.

Unit V

Network Analysis: Drawing Network Diagrams – Critical Path Method – Concept of Slack and Floats on network – PERT – Three types of time estimates – Algorithm for PERT – Differences in PERT and CPM.

UNIT-I

Linear Programming Problem

Linear programming can be used in a variety of situations. In most of the business or economical situation, resources will be limited. The problem there will be to make use of the available resources in such a way that to maximise the production or to minimize the expenditure. These datas can be formulated as linear programming models.

The objective of the linear programming problem is to maximize the profit and minimize the total cost.

The LPP is to determine the values of the decision variables such that all the constraints are satisfied and gives the maximum or minimum value for the objective function. The maximum or minimum value of the objective function is called an Optimum value.

Formulation of LPP:

The formulation of any situation to a LPP is based on the following guidelines:

1. Identification of decision variables
2. Formation of objective function which is to be either maximize or minimize.
3. The various constraints involved due to the limited availability of resources.

Mathematical Formulation of LPP:

The general form of LPP is as follows:

$$Max\ or\ Min\ z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

Subject to, $a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \left(\begin{smallmatrix}\le\\ \ge\\ =\end{smallmatrix}\right) b_1$

$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \left(\begin{smallmatrix}\le\\ \ge\\ =\end{smallmatrix}\right) b_2$$

…………

$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \left(\begin{smallmatrix}\le\\ \ge\\ =\end{smallmatrix}\right) b_m,\ x_i \ge 0, \forall i.$$

Here the variables x_i are called decision variables. The function z is called the objective function and $c_1, c_2 \ldots c_n$ are called the cost variables.

The linear constraints of the general LPP involved one of the relations," = or $\leq or \geq$". The addition of some non-negative variables on the left hand side of the inequations may convert inequations into equations. For that purpose we use the variables called slack or surplus variables.The variable which we add to constraint equation is called slack variable. The variable which we subtract to constraint equation is called surplus variable.

An n-tuple$(x_1, x_2 \ldots x_n)$ of real numbers which satisfies the linear constraints and the non-negative restriction of the given LPP is called feasible solution.

An n-tuple$(x_1, x_2 \ldots x_n)$ of real numbers which satisfies the linear constraints and the non-negative restriction of the given LPP and optimize the objective function is called optimum feasible solution.

A basic solution to the system AX=B is called degenerate if one or more of the basic variables are zero.

Graphical Method

If the objective function z is a function of two variables then the problem can be solved by graphical method. The procedure is as follows.

Step 1: First of all we consider the constraints as equalities or equations.

Step 2: Then we draw the lines in the plane corresponding to each equation obtained in step 1 and non-negative restrictions.

Step 3: Then we find the permissible region(region which is common to all the equations) for the values of the variables which is the region bounded by the lines drawn in step 2.

Step 4: Finally we find a point in the permissible region which gives the optimum value of the objective function.

Note:

1. If there is no permissible region in a problem then we say that the problem has no solution using graphical method.

2. If the maximum value of z appears only at ∞ then the problem has unbounded solution.
3. If the maximum value of z appears at 2 points there exists infinite number of solutions to the LPP.

Problem:

Find the maximum value of z to the following LPP using graphical method, $Max\ z = 5x_1 + 7x_2$, subject to $x_1 + x_2 \leq 4, 3x_1 + 8x_2 \leq 24,$

$$10x_1 + 7x_2 \leq 35\ and\ x_i \geq 0, \forall\ i = 1,2.$$

Solution:

The given problem contains only two variables $x_1\ and\ x_2$. So graphical method is possible. Consider the equations, we get

$$x_1 + x_2 = 4,$$

$$3x_1 + 8x_2 = 24,$$

$$10x_1 + 7x_2 = 35$$

Now to find the points,

Consider $x_1 + x_2 = 4,$

Put $x_1 = 0 => x_2 = 4$

and $x_2 = 0 => x_1 = 4$

Therefore (0,4) and (4.0) are the pts.

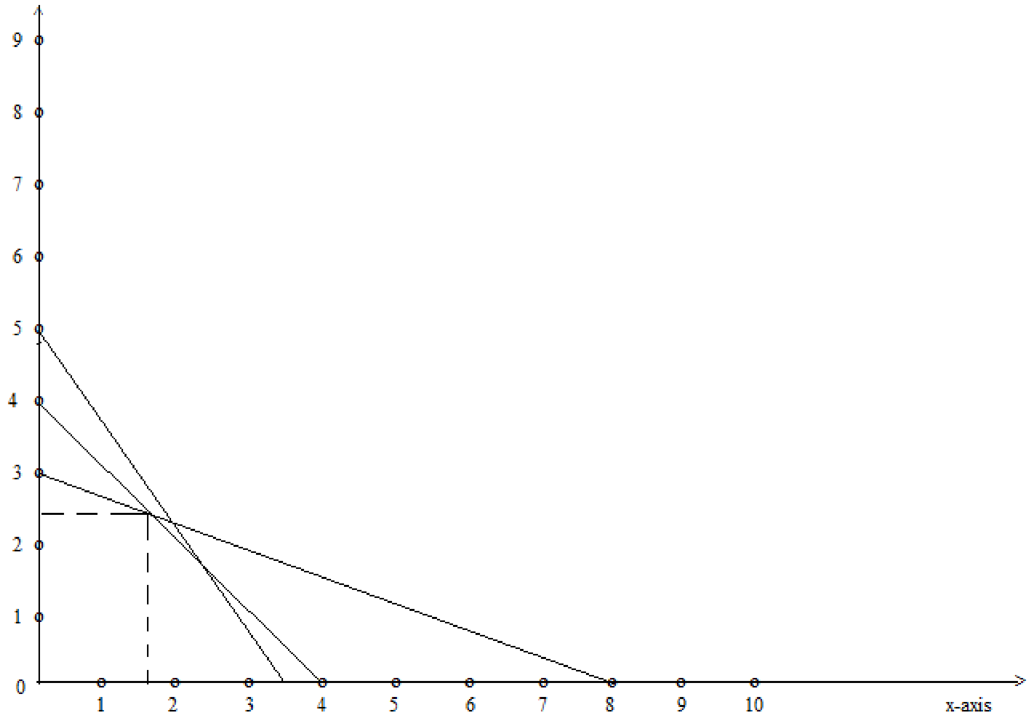
Similarly consider $3x_1 + 8x_2 = 24.$

We get (0,3) and (8,0) are the pts.

Consider $10x_1 + 7x_2 = 35.$

We get (0,5) and (3.5,0) are the pts.

The graphical representation is



Solution is $x_1 = 1.6,\ x_2 = 2.4$ and $Max\ z = 24.8$

Exercise:

1. Solve graphically the following LPP $Max\ z = x_1 + x_2$, subject to $5x_1 + 3x_2 \leq 15, x_1 + x_2 \geq 6,\ x_i \geq 0, \forall\ i = 1,2.$
2. Solve graphically the following LPP $Min\ z = 2x_1 + x_2$, subject to $3x_1 + x_2 \geq 3, 4x_1 + 3x_2 \geq 6, x_1 + 2x_2 \geq 2,\ x_i \geq 0, \forall\ i = 1,2.$

Simplex Method

The procedure is as follows.

Step 1: The objective function of the given LPP is to be maximised or convert into a maximisation problem.

Step 2: All the b_i's should be non-negative. If some b_i is negative multiply the corresponding equation by -1.

Step 3: The inequations of the constraint must be converted into equations by introducing slack or surplus variables in the constraints. The cost of these variables are taken to be zero.

Step 4: Obtain the initial basic feasible solution $X_B = B^{-1}b$ and form the starting simplex table.

Step 5: Compute the net evaluation $z_j - c_j$ (j=1,2,…n), where $z_j = \sum_{j=1}^{n} C_{Bi} Y_{ij}$ (i=1,2..m). If all $z_j - c_j \geq 0$, then the initial basic feasible soln is an optimum basic feasible solution.

Step 6: Choose the most negative $z_j - c_j$. Let it be $z_r - c_r$. If all $Y_{ir} < 0$ then there is an indication of unbounded solution.

Step 7: Compute the ratio $\frac{X_{Bi}}{Y_{ir}}$. The (k,r)th element is called leading element or pivot element.

Step 8: Divide each element of the kth row by the pivot element and make all other elements in the rth column to zero by elementary row operation.

Step 9: Repeat step 5 to 8 until an optimum solution is obtained.

Problem:

Solve the following LPP using simplex method,

$Max\ z = 5x_1 + 3x_2$, subject to $x_1 + x_2 \leq 2, 5x_1 + 2x_2 \leq 10,$

$$3x_1 + 8x_2 \leq 12\ and\ x_i \geq 0, \forall\ i = 1,2.$$

Solution:

The objective function is maximization so the resulting LPP becomes

$Max\ z = 5x_1 + 3x_2 + 0x_3 + 0x_4 + 0x_5$ subject to

$$x_1 + x_2 + x_3 = 2$$

$$5x_1 + 2x_2 + x_4 = 10$$

$$3x_1 + 8x_2 + x_5 = 12, x_i \geq 0, \forall\ i = 1,2\ldots5$$

The initial basic feasible solution is obtained by putting $x_1 = x_2 = 0$ in the reformulated form of LPP and we get $x_3 = 2, x_4 = 10, x_5 = 12$.

<u>Starting Simplex Table:</u>

		C_j	5	3	0	0	0	Ratio
C_B	X_B	Y_B	Y_1	Y_2	Y_3	Y_4	Y_5	$\frac{X_{Bi}}{Y_{ir}}$
0	X_3=2	Y_3	1	1	1	0	0	2
0	X_4=10	Y_4	5	2	0	1	0	2
0	X_5=12	Y_5	2	8	0	0	1	4
	$z_j = \sum_{j=1}^{n} C_{Bi} Y_{ij}$		0	0	0	0	0	
	$z_j - c_j$		-5	-3	0	0	0	

<u>First Iteration:</u>

New Pivot equation=$\frac{old\ \ equation}{pivot\ \ element}$

		C_j	5	3	0	0	0
C_B	X_B	Y_B	Y_1	Y_2	Y_3	Y_4	Y_5
5	X_1=2	Y_1	1	1	1	0	0
0	X_4=0	Y_4	0	-3	-5	1	0
0	X_5=6	Y_5	0	5	-3	0	1
	$z_j = \sum_{j=1}^{n} C_{Bi} Y_{ij}$		5	5	5	0	0
	$z_j - c_j$		0	2	5	0	0

Hence an optimal basic feasible solution is obtained.

Solution is $x_1 = 2,\ x_2 = 0$ and $Max\ z = 10$

Exercise:

1. Solve the following LPP using simplex method,
   $Min\ z = x_1 - 3x_2 + 2x_3$, subject to $3x_1 - x_2 + 2x_3 \le 7$,
   $-2x_1 + 4x_2 \le 12, -4x_1 + 3x_2 + 8x_3 \le 10\ and\ x_i \ge 0, \forall\ i = 1,2,3.$
2. Solve the following LPP using simplex method,

$Max\ z = 25x_1 + 20x_2$, subject to $16x_1 + 12x_2 \leq 100$,
$8x_1 + 16x_2 \leq 80, and\ x_i \geq 0, \forall\ i = 1,2$.

Big-M Method[Method of Penalty]

In the Big-M method huge negative cost (-M) is assigned to the artificial value, while zero cost to the slack and surplus variables in the objective function. The usual simplex method is then followed. At any iteration there can arise any one of the following two cases.

Case(i): There is atleast one vector corresponding to some artificial variable in the basis at zero level and all the net evaluation $z_j - c_j \geq 0$. In this case the given LPP doesnot possess an optimum basic feasible solution.

Case(ii): There is atleast one vector corresponding to some artificial variable in the basis not at zero level and all the net evaluation $z_j - c_j \geq 0$. In this case the given LPP doesnot possess an optimum basic feasible solution.

Whenever an artificial variable leaves from the basis we omit all the entries corresponding to that variable from the simplex table in the next iteration. A drawback of this method is the possible computational error that could result from assigning a very large value to M.

Problem:

Solve the following LPP using Big-M method, $Min\ z = 12x_1 + 20x_2$, subject to $6x_1 + 8x_2 \geq 100$, $7x_1 + 12x_2 \geq 120, and\ x_i \geq 0, \forall\ i = 1,2$.

Solution:

The objective function is converted to maximisation objective function.

The resulting LPP becomes,

$Max(-z) = -12x_1 - 20x_2 + 0x_3 + 0x_4 - Mx_5 - Mx_6$ subject to

$$6x_1 + 8x_2 - x_3 + x_5 = 100$$

$$7x_1 + 12x_2 - x_4 + x_6 = 120, x_i \geq 0, \forall\ i.$$

The initial basic feasible solution is obtained by putting $x_1 = x_2 = x_3 = x_4 = 0$ in the reformulated form of LPP and we get $x_5 = 100, x_6 = 120$.

Starting Simplex Table:

		C_j	-12	-20	0	0	-M	-M	Ratio
C_B	X_B	Y_B	Y_1	Y_2	Y_3	Y_4	Y_5	Y_6	$\frac{X_{Bi}}{Y_{ir}}$
-M	X_5=100	Y_5	6	8	-1	0	1	0	12.5
-M	X_6=120	Y_6	7	12	0	-1	0	1	10
	$z_j = \sum_{j=1}^{n} C_{Bi} Y_{ij}$		-13M	-20M	M	M	-M	-M	
	$z_j - c_j$		-13M+20	-20M+20	M	M	0	0	

First Iteration:

		C_j	-12	-20	0	0	-M	Ratio
C_B	X_B	Y_B	Y_1	Y_2	Y_3	Y_4	Y_5	$\frac{X_{Bi}}{Y_{ir}}$
-M	X_5=20	Y_5	4/3	0	-1	2/3	1	15
-20	X_2=10	Y_2	7/12	1	0	-1/12	0	120/7
	$z_j = \sum_{j=1}^{n} C_{Bi} Y_{ij}$		-4M/3-35/3	-20	M	-2M/5+5/3	-M	
	$z_j - c_j$		-4M/3+1/3	0	M	-2M/5+5/3	0	

Second Iteration:

		C_j	-12	-20	0	0
C_B	X_B	Y_B	Y_1	Y_2	Y_3	Y_4
-12	X_1=15	Y_1	1	0	-3/4	½
-20	X_2=5/4	Y_2	0	1	7/16	-5/12
	$z_j = \sum_{j=1}^{n} C_{Bi} Y_{ij}$		-12	-20	¼	7/3
	$z_j - c_j$		0	0	¼	7/3

Hence an optimal basic feasible solution is obtained.

Solution is $x_1 = 15,\ x_2 = 5/4$ and $Max(-z) = -205$

Therefore $Min(z) = -(-205) = 205$

Exercise:

1. Solve the following LPP using Big-M method, $Max\ z = x_1 + 2x_2$, subject to $2x_1 + x_2 \leq 2, 3x_1 + 4x_2 \geq 12 and\ x_i \geq 0, \forall\ i = 1,2$.
2. Solve the following LPP using Big-M method, $Max\ z = 8x_1 - 4x_2$, subject to $4x_1 + 5x_2 \leq 20, x_1 - 3x_2 \leq 23, and\ x_1 \geq 0, x_2$ is unrestricted.

<u>Two-Phase Simplex Method</u>

Two-phase method consists of two different phases.

Phase(i): In phase (i) we consider the objective objective function min z^*=sum of the artificial variables and cost of the artificial variables are taken to be -1 in the objective function. Here the cost of the given variables, slack variables and surplus variables are taken to be zero. Solve the LPP as usual simplex method. If min z^*=0 then we proceed to phase(ii).

Phase(ii): Final table of phase(i) is initial table of phase(ii) but we consider the original objective function, ie) replace the cost by original cost.

Problem:

Solve the following LPP using Big-M method, $Min\ z = 60x_1 + 80x_2$, subject to $20x_1 + 30x_2 \geq 900$, $40x_1 + 30x_2 \geq 1200, and\ x_i \geq 0, \forall\ i = 1,2$.

Solution:

The objective function is converted to maximisation objective function.

The resulting LPP becomes,

$Max(-z) = -60x_1 - 80x_2 + 0x_3 + 0x_4 - Mx_5 - Mx_6$ subject to

$$20x_1 - 30x_2 - x_3 + x_5 = 900$$

$$40x_1 + 30x_2 - x_4 + x_6 = 1200, x_i \geq 0, \forall\ i.$$

The initial basic feasible solution is obtained by putting $x_1 = x_2 = x_3 = x_4 = 0$ in the reformulated form of LPP and we get $x_5 = 900, x_6 = 1200$.

<u>Starting Simplex Table:</u>

		C_j	-60	-80	0	0	-M	-M	Ratio
C_B	X_B	Y_B	Y_1	Y_2	Y_3	Y_4	Y_5	Y_6	$\frac{X_{Bi}}{Y_{ir}}$
-M	X_5=900	Y_5	20	30	-1	0	1	0	45
-M	X_6=1200	Y_6	40	30	0	-1	0	1	30
	$z_j = \sum_{j=1}^{n} C_{Bi} Y_{ij}$		-60M	-60M	M	M	-M	-M	
	$z_j - c_j$		-60M+60	-60M+80	M	M	0	0	

First Iteration:

		C_j	-60	-80	0	0	-M	Ratio
C_B	X_B	Y_B	Y_1	Y_2	Y_3	Y_4	Y_5	$\frac{X_{Bi}}{Y_{ir}}$
-M	X_5=300	Y_5	0	15	-1	1/2	1	20
-60	X_1=30	Y_6	1	¾	0	-1/40	0	40
	$z_j = \sum_{j=1}^{n} C_{Bi} Y_{ij}$		-60	-15M-45	M	-M/2+3/2	-M	
	$z_j - c_j$		0	-15M+35	M	-M/2+3/2	0	

Second Iteration:

		C_j	-60	-80	0	0
C_B	X_B	Y_B	Y_1	Y_2	Y_3	Y_4
-80	X_2=20	Y_2	0	1	-1/15	1/30
-60	X_1=15	Y_1	1	0	1/20	-1/20
	$z_j = \sum_{j=1}^{n} C_{Bi} Y_{ij}$		-60	-80	7/3	1/3
	$z_j - c_j$		0	0	7/3	1/3

Hence an optimal basic feasible solution is obtained.

Solution is $x_1 = 15,\ x_2 = 20$ and $Max(-z) = -2500$

Therefore $Min(z) = -(-2500) = 2500$

Exercise:

1. Solve the following LPP using Big-M method, $Max\ z = 2x_1 + 3x_2$, subject to $x_1 + x_2 + 2x_3 \leq 5, 2x_1 + 3x_2 + 4x_3 = 12 and\ x_i \geq 0, \forall\ i =$ 1,2,3.
2. Solve the following LPP using Big-M method, $Max\ z = 4x_1 + x_2$, subject to $3x_1 + x_2 = 3, x_1 + 2x_2 \leq 4, 4x_1 + 3x_2 \geq 6, and\ x_1, x_2 \geq 0$.

Duality

Every LPP associated with another LPP is called the dual of the problem. The given problem is called primal problem. If the optimal solution of one problem is known then the optimal solution of other is also known.

Formulation of Dual LPP:

Consider the following LPP in canonical form,

$$Max\ z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

Subject to, $a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \leq b_1$

$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \leq b_2$

…………

$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \leq b_m, x_i \geq 0, \forall i.$

The above problem is called the primal problem and the variables are called primal variables and the constraints are called primal constraints.

The dual of the given primal problem is defined as,

$$Min\ z^* = b_1y_1 + b_2y_2 + \cdots + b_my_m$$

Subject to, $a_{11}y_1 + a_{21}y_2 + \cdots + a_{m1}y_m \geq c_1$

$a_{12}y_1 + a_{22}y_2 + \cdots + a_{m2}y_m \geq c_2$

…………

$a_{1n}y_1 + a_{2n}y_2 + \cdots + a_{mn}y_m \geq c_n, y_i \geq 0, \forall i.$

Here the variables are called dual variables and the constraints are called dual constraints.

Example: Write the dual of the following LPP.

$Max\ z = x_1 + 2x_2$, subject to $2x_1 - 3x_2 \leq 3, 4x_1 + x_2 \leq -4, x_1, x_2 \ \geq 0$.

The dual is,

$Min\ z^* = 3y_1 - 4y_2$, subject to $2y_1 + 4y_2 \geq 1, -3y_1 + y_2 \geq 2, x_1, x_2 \ \geq 0$.

Dual Simplex Method

The dual simplex method is moreover same as the standard simplex method. Here we donot require any artificial variable. Hence a lot of computational work is minimized in this method. The procedure is as follows.

Step 1: Convert the minimization LPP into maximization LPP.

Step 2: Convert all ≥ inequalities into ≤ inequalities by multiplying the corresponding equation by-1.

Step 3: Introduce any slack variables and obtain the starting equation.

Step 4: After finding out the initial basic feasible solution construct the starting simplex table. In this table calculate the net evaluation $z_j - c_j$ and test the nature of $z_j - c_j$. Here the following subcases arises.

Case(i): If all $z_j - c_j \geq 0$ and the X_{Bi} column vectors are ≥ 0 optimum basic feasible solution is obtained.

Case(ii): All $z_j - c_j \geq 0$ and and one of $X_{Bi} < 0$ we go to the bnext stepo.

Case(iii): If one $z_j - c_j < 0$ then the dual simplex method fails.

Step 5: Determine the leaving variable by selecting the most negative X_{Bi} . Then calculate the replacement ratio max $\left(\frac{z_j - c_j}{Y_{kj}}, Y_{kj} < 0\right)$ and determine the entering variable.

Step 6: The leaving row and the entering column intersect at a point and that element is called pivot element.

Step 7: Now test for optimality and repeat the process until the optimum basic feasible solution is obtained or there is an indication of no feasible solution.

Problem:

Solve the following LPP using Dual Simplex method, $Min\ z = 3x_1 + x_2$, subject to $x_1 + x_2 \geq 1, 2x_1 + 3x_2 \geq 2, x_1, x_2 \ \geq 0$.

Solution:

The objective function is a minimization problem so convert it into a maximisation problem. The resulting LPP becomes,

$Max\ z^* = -3x_1 - x_2 + 0x_3 + 0x_4$ subject to

$$-x_1 - x_2 + x_3 = -1$$

$$-2x_1 - 3x_2 + x_4 = -2$$

$$x_i \geq 0, \forall\ i = 1,2,3,4.$$

The initial basic feasible solution is obtained by putting $x_1 = x_2 = 0$ in the reformulated form of LPP and we get $x_3 = -1, x_4 = -2$.

Starting Simplex Table:

		C_j	-3	-1	0	0
C_B	X_B	Y_B	Y_1	Y_2	Y_3	Y_4
0	X_3=-1	Y_3	-1	-1	1	0
0	X_4=-2	Y_4	-2	-3	0	1
	$z_j = \sum_{j=1}^{n} C_{Bi} Y_{ij}$		0	0	0	0
	$z_j - c_j$		3	1	0	0

For calculating the entering variable we find out the replacement ratio

By max $\left(\frac{z_j - c_j}{Y_{kj}}, Y_{kj} < 0\right)$=max(-3/2, -1/3, 0/0, 0/1)= -1/3.

First Iteration:

		C_j	-3	-1	0	0
C_B	X_B	Y_B	Y_1	Y_2	Y_3	Y_4

0	X_3=-1/3	Y_3	-1/3	0	1	-1/3
-1	X_2=2/3	Y_2	2/0	1	0	-1/3
	$z_j = \sum_{j=1}^{n} C_{Bi} Y_{ij}$		-2/3	-1	0	1/3
	$z_j - c_j$		7/3	0	0	1/3

Calculate the ratio max($\frac{z_j - c_j}{Y_{kj}}, Y_{kj} < 0$)=max(-7, 0/0, 0/1, -1)= -1.

<u>Second Iteration:</u>

		C_j	-3	-1	0	0
C_B	X_B	Y_B	Y_1	Y_2	Y_3	Y_4
0	X_4=1	Y_4	1	0	-3	1
-1	X_2=1	Y_2	-1/3	1	-1	0
	$z_j = \sum_{j=1}^{n} C_{Bi} Y_{ij}$		1/3	-1	1	0
	$z_j - c_j$		10/3	0	1	0

Hence an optimal basic feasible solution is obtained.

Solution is $x_1 = 0, x_2 = 1$ and $Max\ z^* = -1$

Min z= - $Max\ z^*$= -(-1)=1

****

UNIT-II

Transportation Problems

Transportation problem is one of the subclasses of Linear Programming problem and can be regarded as the generalization of Assignment problem. The TP leads with the determination of minimum cost plan for transporting a single commodity from a number of sources to a number of destinations.

Mathematical Formulation

Let $O_1, O_2, \ldots, O_m$ be the m origins and $D_1, D_2, \ldots, D_n$ be the n destinations. Let a_i be the amount of supply at the origin and b_j be the demand at destination. Let C_{ij} be the unit transportation cost.

The transportation problem is, Min z=$\sum\sum C_{ij}\, X_j$

Subject to $\sum_j x_{ij} \le a_i$,

$\sum_j x_{ij} \ge b_j$

If the total supply equal to the total demand then the TP is called balanced TP. Otherwise it is unbalanced. In this case we have either $\sum a_i > \sum b_j$ or $\sum a_i < \sum b_j$.

If the transportation table have mn unknowns there are m+n constraints. These m+n equations are not independent. Hence there are m+n-1 independent constraints.

A feasible solution to a TP is a set of non-negative individual allocations which satisfies the row and the column.

A feasible solution of a TP is said to be a basic feasible solution if the total number of positive allocations is exactly equal to m+n-1.

A feasible solution is said to be optimum if it minimizes as the total transportation cost. If a feasible solution involves exactly m+n-1 independent individual positive allocations, then it is known as non-degenerate basic feasible solution.

Determination of Feasible Solution of a TP:

1. North West Corner Rule

The procedure is as follows:

Step 1: Start with the cell (1,1) at the north-west corner of the transportation table and allocate it maximum possible amount x_{11}=min(a_1,b_1) so that either the capacity of the origin O_1 or the requirement for the destination D_1 satisfies it or both.

Step 2: If $x_{11}=a_1$, cross off the first row of the transportation table and decrease b_1 by a_1 and go to step 3. If $x_{11}=b_1$, cross off the first column of the transportation table and decrease a_1 by b_1 and go to step 3. If $x_{11}=a_1= b_1$, cross off either the first row or the first column(but not both).

Step 3: Repeat steps 1 and 2 for the resulting reduced transportation table until all the requirements are satisfied.

Note:

The N-W corner rule of allocating values of the variables eliminates either a row or column of the table from further consideration and the last allocation eliminates both a row and column. Hence a feasible solution obtained by this method has m+n-1 basic variables.

Problem:

Find an initial basic feasible solution to the following TP using North-West Corner Rule.

	1	2	3	4	a_i
1	10	0	20	11	15
2	12	7	9	20	25
3	0	14	16	18	5
b_j	5	15	15	10	45

Solution:

Step 1: First allocate in (1,1) cell. x_{11}=min(15,5)=5

	1	2	3	4	a_i
1	10	0	20	11	15
2	12	7	9	20	25
3	0	14	16	18	5
b_j	5	15	15	1	45

Step 2: Second allocate in (1,2) cell. x_{12}=min(10,15)=10

	2	3	4	
~~1~~	~~0~~	~~20~~	~~11~~	~~10~~
2	7	9	20	25
3	14	16	18	5
	5	15	10	

Step 3: Third allocate in (2,2) cell. X_{22}=min(25,5)=5

	2	3	4	
2	7	9	20	25
3	14	16	18	5
	5	15	10	

Step 4: Fourth allocate in (2,3) cell. X_{23}=min(20,15)=15

	3	4	
2	9	20	20
3	16	18	15
	15	10	

Step 5: Fifth allocate in (2,4) cell. X_{24}=min(5,10)=5

	4	
~~2~~	~~20~~	~~5~~
3	18	5
	10	

Step 6: Sixth allocate in (3,4) cell. X_{34}=min(5,5)=5

	4	
3	18	5
	5	

The initialz basic feasible solution is given by $x_{11} = 5, x_{12} = 10, x_{22} = 5,$

$$x_{23} = 15, x_{24} = 5, x_{34} = 5$$

	1	**2**	**3**	**4**
1	**5** **10**	**10** **0**	 **20**	 **11**
2	 **12**	**5** **7**	**15** **9**	**5** **20**
3	 **0**	 **14**	 **16**	**5** **18**

Minimum transportation cost=5*10+10*0+5*7+15*9+5*20+5*18=Rs.410.

Exercise:

1. Using North West Corner rule find out the initial basic feasible solution to the following TP.

	1	2	3	4	a_i
1	6	4	1	5	14
2	8	9	2	7	16
3	4	3	6	2	5
b_j	6	10	15	4	35

2. Least-Cost Method

The procedure is as follows:

Step 1: Determine the smallest cost in the cost matrix of the transportation table. Let it be C_{ij}. Allocate $x_{ij}=\min(a_i,b_j)$ in the (i,j) cell.

Step 2: If $x_{ij}=a_i$ cross off i^{th} row of the transportation table and decrease b_j by a_i. If $x_{ij}=b_j$ cross off j^{th} column of the transportation table and decrease a_i by b_j. If $x_{ij}=a_i= b_j$, cross off either the first row or the first column(but not both).

Step 4: Repeat steps 1 and 2 for the resulting table. Reduce transportation table until all requirements are satisfied.

Problem:

Find an initial basic feasible solution to the following TP using Least Cost Method.

	1	2	3	4	
1	10	0	20	11	15
2	12	7	9	20	25
3	0	14	16	18	5
	5	15	15	10	

Solution:

Step 1: 0 is the minimum element which appears in (1,2) and (3,1) cells. For the First allocation in $x_{12}=\min(15,15)=15$. Cross out first row or first column.

	1	2	3	4	
~~1~~	~~10~~	~~0~~	~~20~~	~~11~~	~~15~~
2	12	7	9	20	25
3	0	14	16	18	5
	5	15	15	1	

Step 2: Second allocate in (3,1) cell. X_{31}=min(5,5)=5

	~~1~~	2	3	4	
2	~~12~~	7	9	20	25
3	~~0~~	14	16	18	5
	~~5~~	0	15	10	

Step 3: Third allocate in (2,2) cell. X_{22}=min(25,0)=0

	~~2~~	3	4	
2	~~7~~	9	20	25
3	~~14~~	16	18	0
	~~0~~	15	10	

Step 4: Fourth allocate in (2,3) cell. X_{23}=min(25,15)=15

	~~3~~	4	
2	~~9~~	20	25
3	~~16~~	18	5
	~~15~~	10	

Step 5: Fifth allocate in (3,4) cell. X_{34}=min(0,10)=0

	4	
2	20	10
~~3~~	~~18~~	~~0~~
	10	

Step 6: Sixth allocate in (2,4) cell. X_{24}=min(10,10)=10

	4		
2	20	10	
	10		

The initial basic feasible solution is given by $x_{12} = 15, x_{31} = 5, x_{22} = 0,$

$$x_{23} = 15, x_{34} = 0, x_{24} = 10$$

	1	**2**	**3**	**4**
1	 **10**	**15** **0**	 **20**	 **11**
2	 **12**	**0** **7**	**15** **9**	**10** **20**
3	**5** **0**	 **14**	 **16**	**0** **18**

Minimum transportation cost=15*9+10*20=Rs.335.

Exercise:

1. Using Least Cost Method find out the initial basic feasible solution to the following TP.

	A_1	A_2	A_3	
P_1	50	30	220	1
P_2	90	45	170	3
P_3	250	250	50	4
	4	2	2	8

3. <u>Vogel's Approximation Method(VAM)</u>

The procedure is as given below.

Step 1: For each row of the transportation table find the smallest cost and next to it. Determine the difference between them for each row. Write them within brackets along the side of the table.

Step 2: For each column do the same step as in step 1.

Step 3: Identify the row or column with the largest difference. Let the greatest difference correspond to p_i and let C_{ij} be the smallest cost in the i^{th} row. Allocate $x_{ij}=\min(a_i,b_j)$ in that cell. If $x_{ij}=a_i$ cross off i^{th} row of the transportation table and decrease b_j by a_i. If $x_{ij}=b_j$ cross off j^{th} column of the transportation table and decrease a_i by b_j. If $x_{ij}=a_i= b_j$, cross off either the first row or the first column(but not both).

Step 4: Repeat steps 1,2 and 3 for the resulting table. Reduce transportation table until all requirements are satisfied.

Problem:

Find an initial basic feasible solution to the following TP using Vogel's Approximation Method.

	1	2	3	4	
1	10	0	20	11	15
2	12	7	9	20	25
3	0	14	16	18	5
	5	15	15	10	

Solution:

Step 1: Here the maximum difference is 14 which corresponds to third row of the table. For the First allocation in $x_{31}=\min(5,5)=5$. Cross out third column or first column.

	1	2	3	4	
1	10	0	20	11	15 (10)
2	12	7	9	20	25 (2)
3	0	14	16	18	5 (14)
	5	15	15	1	
	(10)	(7)	(7)	(7)	

Step 2: Here the maximum difference is 11 which corresponds to first row. Second allocate in (1,2) cell. X_{12}=min(15,15)=15

	2	3	4	
1	0	20	11	15(11)
2	7	9	20	25(2)
3	14	16	18	0 (2)
	15	15	10	
	(7)	(7)	(7)	

Step 3: Here the maximum difference is 11 which corresponds to second column. Third allocate in (2,3) cell. X_{23}=min(25,15)=15

	3	4	
1	20	11	0 (9)
2	9	20	25(11)
3	16	18	0 (2)
	15	10	
	(7)	(7)	

Step 4: Here the maximum difference is 7 which corresponds to fourth column. Fourth allocate in (1,4) cell. X_{14}=min(0,10)=0

	4	
~~1~~	~~11~~	~~0 (11)~~
2	20	10 (20)
3	18	0 (18)
	10	
	(7)	

Step 5: Here the maximum difference is 2 which corresponds to fourth column. Fifth allocate in (3,4) cell. X_{34}=min(0,10)=0

	4	
2	20	10(11)
~~3~~	~~18~~	~~0 (18)~~
	10	
	(2)	

Step 6: Sixth allocate in (2,4) cell. X_{24}=min(10,10)=10

	4	
2	20	10(11)
	10	
	(11)	

The initial basic feasible solution is given by $x_{12} = 15, x_{23} = 15, x_{31} = 5,$

$$x_{14} = 0, x_{34} = 0, x_{24} = 10$$

	1	**2**	**3**	**4**
1	**10**	**15** / **0**	**20**	**0** / **11**
2	**12**	**7**	**15** / **9**	**10** / **20**
3	**5** / **0**	**14**	**16**	**0** / **18**

Minimum transportation cost=15*9+10*20=Rs.335.

Exercise: Using VAM Method find out the initial basic feasible solution to the following TP.

	1	2	3	4	
1	3	1	7	4	300
2	2	6	5	9	400

3	8	3	3	2	500
	250	350	400	200	

Assignment Problems

Assignment problem is a special type of a LPP in which the objective is to assign a number of origins to the equal number of distinct destination at a minimum cost to find the optimum allocation of a number of jobs to an equal number of persons.

Mathematical Formulation of AP

Let there be n machines and n jobs. A job i when processed by machine j at a cost C_{ij}.

$$x_{ij} = \begin{cases} 1, \text{if the ith job is assigned to jth machine} \\ 0, \text{otherwise.} \end{cases}$$

The total assignment cost is $\sum_{i=1}^{n} \sum_{j=1}^{n} C_{ij} X_{ij}$

Hence the AP can be stated mathematically as follows so as to

Minimize z=$\sum_{i=1}^{n} \sum_{j=1}^{n} C_{ij} X_{ij}$

Subject to the constraints $\sum_{i=1}^{n} X_{ij} = \sum_{j=1}^{n} X_{ij} = 1$

Method of Solution

The procedure for the solution of an AP is known as Hungarian Assignment Method is as follows.

Step 1: Subtract the minimum cost of each row of the cost matrix from all the elements of the respective row and get the resultant matrix.

Step 2: Subtract the minimum cost of each column of the cost matrix from all the elements of the respective column and get the resultant matrix. This gives a new cost matrix.

Step 3: In the new cost matrix draw minimum possible number of horizontal and vertical lines to cover all the zeroes. Here two cases arises.

Case (i): If the number of lines=order of cost matrix, optimum assignment has been obtained. Find out the optimum solution.

Case (ii): If the number of lines<order of cost matrix, we go to step-4.

Step 4: Determine the smallest cost in the new cost matrix not covered by the lines. Subtract this cost from all the surviving elements(uncovered by lines) of the new cost matrix and add the same to all those elements of the new cost matrix that are lying at the intersection of horizontal and vertical lines.

Step 5: Repeat steps 3 and 4 until we get p=n(where p denote the number of lines and n denote order of the cost matrix).

Step 6: For determining the optimum assignment consider only the zero elements of the final table. Examine successively the rows(or)columns of the matrix. To find out one with exactly one zero and encircled this zero and mark a cross in the remaining zeroes of the row or column.

Step 6: The assignment schedule corresponding to these zeroes is the optimum assignment schedule and find out the minimum assignment.

Maximisation in AP

In the objective function of the given AP is a maximizing time we converted into minimizing time as in the case of TP by selecting the largest element among all the elements of the cost matrix and then subtracting it from all other elements of the cost matrix. This can equivalently be done by multiplying all the entries of the cost matrix by -1 and then proceeding as in the case of minimizing the AP.

Problem:

A shop has 4 workers and 4 works to be performed. The estimate of time(man hours) each man will take to perform each project is given in the following table. How should the works to be allotted so as to optimise the total work.

	1	2	3	4
1	8	26	17	11
2	13	28	4	26
3	38	19	18	15

$$\begin{array}{c} 4 \end{array} \begin{bmatrix} 19 & 26 & 24 & 10 \end{bmatrix}$$

Solution:

Step 1: Given matrix is,

$$\begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 8 & 26 & 17 & 11 \\ 2 & 13 & 28 & 4 & 26 \\ 3 & 38 & 19 & 18 & 15 \\ 4 & 19 & 26 & 24 & 10 \end{array}$$

Step 2: Here the given matrix is a square matrix. Hence the assignment problem is a balanced problem. Subtracting minimum cost of each row of the cost matrix

we obtain the following matrix.

$$\begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 18 & 9 & 3 \\ 2 & 9 & 24 & 0 & 22 \\ 3 & 23 & 4 & 3 & 0 \\ 4 & 9 & 16 & 14 & 0 \end{array}$$

Step 3: Subtracting minimum cost of each column of the modified cost matrix we obtain the following matrix.

$$\begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 14 & 9 & 3 \\ 2 & 9 & 20 & 0 & 22 \end{array}$$

3	23	0	3	0
4	9	12	14	0

Step 4: Drawing the minimum possible number of horizontal and vertical lines.

	1	2	3	4
1	0	14	9	3
2	9	20	0	22
3	23	0	3	0
4	9	12	14	0

Here the number of lines=4=order of cost matrix.Hence optimum assignment obtained.

	1	2	3	4
1	0			
2			0	
3		0		
4				0

Optimum Schedule is 1->1, 2->3, 3->2, 4->4

Minimum assignment cost=8+4+19+10=41.

Exercise:

1. A company has 5 machines and 5 jobs to be done. The return in rupees of assigning n-machines to n-job as follows. Assign the 5 jobs to the 5 machines so as to maximise the total profit.

	1	2	3	4	5
A	5	11	10	12	4
B	2	4	6	2	5
C	3	12	5	14	6
C	6	14	4	11	7
D	7	9	8	12	5

****

UNIT III

SEQUENCING

Sequencing problem consists of selection of an appropriate order for a finite number of jobs to be done on a finite number of machines/ service facilities in some prescribed order so as to minimize the total idle time for the machines. The problem may have some constraints placed on it, such as due date for each job, processing order of each variable or variable processing times.

Assumptions on sequencing problems

The following simplifying assumptions are usually made for sequencing problems:

1) *Each job, once started on a machine, is to be performed up to completion on that machine.*
2) *The processing time on each machine is known. Such a time is independent of the order of the jobs in which they are to be processed.*
3) *The time taken by each job in changing over from one machine to another machine is negligible.*
4) *A job gets started on a machine as soon as the job and the machine are both idle.*
5) *No machine may process more than one job simultaneously.*
6) *The order of completion of job has no significance (i.e) no job is to be given priority.*

Processing n jobs in 2 machines:

Suppose there are n jobs $J_1, J_2, \ldots\ldots\ldots. J_n$ which are to be processed in two machines say M_1 and M_2 in order $M_1 M_2$ (M_1 first and M_2 next).

Let t_{ij} be the processing time for i^{th} job in j^{th} machine. The list of jobs along with their processing times can be summarized as in the following table

Jobs	J_1	J_2	$\ldots\ldots\ldots\ldots$	J_n
Processing time in M_1	t_{11}	t_{12}	$\ldots\ldots\ldots\ldots$	t_{1n}
Processing time in M_2	t_{21}	t_{22}	$\ldots\ldots\ldots\ldots$	t_{2n}

A job is assigned to machine M_1 first and after it has been completely processed in machine M_1 it is assigned to the machine M_2 . If the machine M_2 is not free at any moment for processing a particular job then that job has to wait in a waiting line for its turn on the machine M_2. In other words passing is not allowed. Hence machine M_1 will always be busy and will process the n jobs one by one. After processing all the n jobs, the machine M_1 remains idle until all the n jobs are completed in the machine M_2 . However M_2 may remain idle after the completion of some of the m jobs and before starting the next job. The sequencing problem is to minimize the total idle time of the second machine M_2.

Let x_{2j} be the time for which the machine M_2 remains idle after completing the $(j-1)^{th}$ job and before starting j^{th} job.

Hence the total idle time for machine M_2 is $\boldsymbol{\sum_{j=1}^{n} x_{2j}}$. Thus the sequencing problem is to minimize $\boldsymbol{\sum_{j=1}^{n} x_{2j}}$. The total elapsed time T is given by

T=Processing time + idle time. (i.e) $T = \sum_{j=1}^{n} t_{2j} + \sum_{j=1}^{n} x_{2j}$.

Here some of the $x_{2j}{}^{s}$ may be zeros. We observe that $\sum_{j=1}^{n} t_{2j}$ is constant.

Hence minimizing T is equivalent to minimizing $\sum_{j=1}^{n} x_{2j}$.

Algorithm to find the optimum sequence for n jobs in 2 machines

Step 1. *List the jobs along with their processing times in a table as given above.*

Step 2. *Find the minimum* (t_{1j}, t_{2j}) *for all* $j = 1,2, \ldots, n$.

Step 3. *If the smallest processing time is for the first machine* M_1 *then place the corresponding job in the first available position in the sequence . If it is for the second machine* M_2 *then place the corresponding job in the last available position in the sequence.*

Step 4. *If there is a tie in the minimum of all the processing times then there arises three cases.*

Case i. *Minimum among all processing times is same for the two machines (i.e) minimum* $t_{1j}, t_{2j}) = t_{1r} = t_{2s}$ *then place the* r^{th} *job in the first available position in the sequence and the* s^{th} *job in the last available position in the sequence.*

Case ii. *If the tie is for the minimum among the processing times* t_{ij} *on machine* M_1 *only, then place the jobs arbitrarily, one after the other, in the last available positions in the sequence.*

Case iii. *If the tie is for the minimum among the processing times* t_{2j} *on machine* M_2 *only, then place the jobs arbitrarily, one after the other, in the last available positions in the sequence.*

Step 5. *Remove the assigned jobs from the table. If the table becomes empty the optimum sequencing is got and stop the procedure. Other wise go to step 2.*

Note. *From the optimal sequencing, calculate the total elapsed time, idle time on machine* M_1 *and idle time on machine* M_2.

(a) ***Total elapsed time*** = *Total time between the starting the first job of the optimum sequence on machine* M_1 *and completing the last job on machine* M_2.
(b) ***Idle time for*** $\boldsymbol{M_1}$ = *Total elapsed time –(Time when the last job in the sequence finishes on* M_1*)*
(c) ***Idle time for*** $\boldsymbol{M_2}$ = *Time at which the first job in the sequence finishes on* $M_1 + \sum_{j=2}^{n}$*(time when the* j^{th} *job in the sequence starts on* M_2*)*$-$*(time when the* $(j-1))^{th}$ *job in the sequence finishes on* M_2*).*

Problem:

Determine the optimum sequence for the following sequencing problem in which 5 jobs are done in 2 machines M_1 *and* M_2 *in the order* $M_1\ M_2$. *Processing times are given in hours in the following table.*

	M_1	M_2
$\boldsymbol{J_1}$	10	4
$\boldsymbol{J_2}$	$\underline{\mathbf{2}}$	12
$\boldsymbol{J_3}$	18	14
$\boldsymbol{J_4}$	6	16
$\boldsymbol{J_5}$	20	6

Solution:

The minimum processing time among all the jobs in the 2 machines M_1 and M_2 is 2 (which is marked in under lined bold face), which correspond to the job J_2 on machine M_1. Thus the job J_2 is placed (sequenced) in the first position in the sequence .

J_2				

We are now left with 4 jobs and their processing times are given below.

Machines / Jobs	$\boldsymbol{M_1}$	$\boldsymbol{M_2}$
$\boldsymbol{J_1}$	10	$\underline{\mathbf{4}}$
$\boldsymbol{J_3}$	18	14
$\boldsymbol{J_4}$	6	16
$\boldsymbol{J_5}$	20	6

Among those four jobs the minimum processing time between the two machines is ***4*** *which corresponds to the job J_1 on machine M_2 . Hence the job J_1 is placed in the last position in the sequence shown below.*

J_2				J_1

The remaining 3 jobs and their processing times are given below.

Machines / Jobs	$\boldsymbol{M_1}$	$\boldsymbol{M_2}$
$\boldsymbol{J_3}$	18	14

$\boldsymbol{J_4}$	**6**	16
$\boldsymbol{J_5}$	20	**6**

The minimum processing time among these jobs is **6** *which corresponds to the job J_4 on machine M_1 and job J_5 on machine M_2. Hence J_4 is placed in the first available position in the sequence as shown below.*

J_2	J_4		J_5	J_1

Finally, the job J_3 is placed as shown below.

J_2	J_4	J_3	J_5	J_1

Thus $J_2 \Longrightarrow J_4 \Longrightarrow J_3 \Longrightarrow J_5 \Longrightarrow J_1$ is the optimal sequence of the five jobs on the two machines.

Sequencing n Jobs on 3 Machines

Processing n Jobs in m Machines

Let there be n jobs $J_1, J_2, \ldots\ldots J_n$ each of which is to be processed in m machines say $M_1, M_2, \ldots\ldots M_m$ in the order $M_1\ M_2 \ldots\ldots M_m$. The list of job numbers 1,2,.....,n with their processing time is given in the following table.

Machines	Job numbers				
	1	2	3	………	n
M_1	t_{11}	t_{12}	t_{13}	………	t_{1n}
M_2	t_{21}	t_{22}	t_{23}	………	t_{2n}
M_3	t_{31}	t_{32}	t_{33}	………	t_{3n}
………	………	………	………	………	………
M_m	t_{m1}	t_{m2}	t_{m3}	………	t_{mn}

An optimum solution to this problem can be obtained if either or both of the following conditions are satisfied.

(a) $\min t_{1j} \geq \max t_{ij}$ for $i = 2,3,\ldots\ldots, k-1$

(b) $\min t_{mj} \geq \max t_{ij}$ for $i = 2,3,\ldots\ldots, k-1$

If the above condition is satisfied then the problem can be converted to an equivalent two machines and n jobs problem.

Algorithm for optimum sequence for n jobs in m machines

The iterative procedure for determining the optimal sequence for n jobs in m machines is given below.

Step 1. *Find* $\min t_{ij}$ *and* $\min t_{mj}$ *. Also find the maximum of each of* t_{2j} *,*t_{3j} *,....*$t_{\overline{k-1}\ j}$ *for all* $j = 1,2,\ldots,n$.

Step 2. *Check the following*

(a) $\min t_{1j} \geq \max t_{ij}$ *for* $i = 2,3,\ldots,k-1$
(b) $\min t_{mj} \geq \max t_{ij}$ *for* $i = 2,3,\ldots,k-1$

Step 3. *If both the inequalities of step 2 are not satisfied this method fails otherwise go to step 4.*

Step 4. *Convert the m machines problem into a two machine problem by introducing two fictious machines H and K such that*

$t_{Hj} = t_{1j} + t_{2j} + \cdots + t_{\overline{k-1}\ j}$ *and*

$t_{Kj} = t_{2j} + t_{3j} + \cdots + t_{kj}$

Where t_{Hj} *is the processing time of* j^{th} *job in machine H and* t_{Kj} *is the processing time of* j^{th} *job in machine K.*

Step 5. *Determine the optimal sequence for n jobs and 2 machines sequencing problem .*

Problem :

Find the sequence that minimizes the total elapsed time required to complete the following six jobs on three machines M_1, M_2 *and* M_3 *in the order* $M_1 \Longrightarrow M_2 \Longrightarrow M_3$

Jobs⟹		J_1	J_2	J_3	J_4	J_5	J_6
Machines⟹	M_1	3	12	5	2	9	11
	M_2	8	6	4	6	3	1
	M_3	13	14	9	12	8	13

Solution:

We are given 6 jobs each of which is to be processed in 3 machines $\boldsymbol{M_1}$ *,* $\boldsymbol{M_2}, \boldsymbol{M_3}$ *in the order* $\boldsymbol{M_1}\ \boldsymbol{M_2}\ \boldsymbol{M_3}$ *. Hence n=6 and m=3. Hence the given problem becomes 6 jobs in 3 machines problem. For optimum sequence the required condition is either or both of the following conditions must be true.*

$$(a)\, min\, t_{1j} \geq max\, t_{2j} \quad j = 2, \ldots\ldots, 5$$
$$(b)\, min\, t_{3j} \geq max\, t_{2j} \quad j = 2, \ldots\ldots, 5$$

	$\boldsymbol{M_1}$	$\boldsymbol{M_2}$	$\boldsymbol{M_3}$
Minimum time	*2*	*1*	*8*
Maximum time	*12*	*8*	*14*

$Min\ t_{1j} = 2$ *and* $n\ t_{3j} = 8$ *;* $Max\ t_{2j} = 8$

Clearly $Min\ t_{3j} = Max\ t_{2j} = 8$ *. Hence the condition to reduce 3 machines problem to 2 maxhines problem is satisfied. Hence the given problem can be converted to an equivalent two machines for six jobs problem.*

Let H and K be two fictious machines such that the processing times are got from the following relations

$H_i = M_{i1} + M_{i2}(i.e.)H_i = t_{Hj} = t_{1j} + t_{2j}$

$K_i = M_{i2} + M_{i3}(i.e.)K_i = t_{Kj} = t_{2j} + t_{3j}$

Where $1 \leq i \leq 6$.

Hence we have the processing table for the two machines H and K.

Jobs⟹		J_1	J_2	J_3	J_4	J_5	J_6
Machines⟹	***H*** $(\boldsymbol{M_1} + \boldsymbol{M_2})$	*11*	*18*	*9*	*8*	*12*	*12*
	K $\boldsymbol{M_2} + \boldsymbol{M_3})$	*21*	*20*	*13*	*18*	*11*	*14*

Now we find the optimum sequence of the 6 jobs in two machines H and K as usual in the following tables.

J_4					
J_4	J_3				

J_4	J_3	J_1			J_5

J_4	J_3	J_1	J_6		J_5

J_4	J_3	J_1	J_6	J_2	J_5

Hence the optimal sequencing is $J_4 \Longrightarrow J_3 \Longrightarrow J_1 \Longrightarrow J_6 \Longrightarrow J_2 \Longrightarrow J_5$.

Table to find minimum elapsed time

Job seq	M_1 *Time in Time out*		M_2 *Time in Time out*		M_3 *Time in Time out*		*Idle time* M_2	*Idle time* M_3
J_4	*0*	*2*	*2*	*8*	*8*	*20*	*2*	*8*
J_3	*2*	*7*	*8*	*12*	*20*	*29*	-	*0*
J_1	*7*	*10*	*12*	*20*	*29*	*42*	-	*0*
J_6	*10*	*21*	*21*	*22*	*42*	*55*	*1*	*0*
J_2	*21*	*33*	*33*	*39*	*55*	*69*	*11*	*0*
J_5	*33*	***42***	*42*	***45***	*69*	***77***	*3*	*0*
Total		***17***	***8***					

Minimum total elapsed time = 77 Hrs

Idle time for M_1*=77-42=35*

Idle time for M_2*=17+(77-45)=17+32=49*

Idle time for M_3=8.

Processing 2 jobs in m machines

Suppose there are 2 jobs J_1, J_2 which are processed in m machines M_1, $M_2, \dots M_m$ in two different orders. The ordering of each of the two jobs in m machines in known in advance. The ordering may not be the same for both the jobs. Also the processing times for each of the two jobs are also known. Let t_{ij} be the processing time of job J_i in machine M_j where i=1,2 and j=1,2,…,m. The optimal sequence can be obtained by the following graphical method. The solution procedure is summarized in the following steps.

Step 1. *Draw two perpendicular lines, horizontal and vertical. Horizontal line represents the processing time for J_1 while the job J_2 remains idle. Vertical line represents the processing time for job J_2 while the job J_1 remains idle.*

Step 2. *Mark the processing time for jobs J_1 and J_2 on the horizontal and vertical lines respectively according to the given order of machines.*

Step 3. *Construct various blocks starting from the orgin (starting point) by pairing the same machines until the end point.*

Step 4. *Draw the line starting from the origin to end point by moving horizontally, vertically and diagonally along a line which makes an angle of $45°$ with the horizontal line (base). The horizontal segment of this line indicates that first job is under process while the second job is idle. Similarly, the vertical segment of this line indicates that the second job is under process while the first job is idle. The diagonal segment of the line shows that both the jobs are under process simultaneously.*

Step 5. *An optimum path is one that minimizes the idle time for both the jobs. Thus we must choose the path on which diagonal movement is maximum.*

Step 6. *The total elapsed time is obtained by adding the idle time for either job to the processing time for that job.*

The total elapsed time =Processing time for job J_1+ idle time for J_1

Or

The total elapsed time =Processing time for job J_2+ idle time for J_2

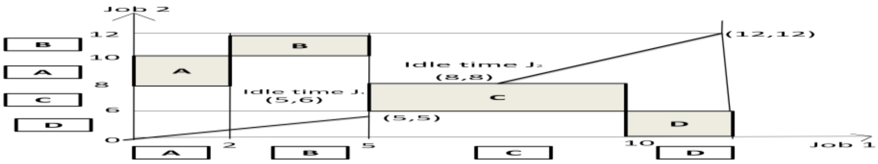
Problem:

Use graphical method to find the minimum total elapsed time needed to process the following two jobs on four machines A,B,C,D given the processing times and the sequences.

Job 1	*Sequence:* *Time in Hrs:*	*A* *2*	*B* *3*	*C* *5*	*D* *2*
Job 2	*Sequence:* *Time in Hrs:*	*D* *6*	*C* *2*	*A* *3*	*B* *1*

Solution:

The graph representing the data is given below. The rectangular blocks in the graph are obtained by pairing the same machines. The path from the origin to the end point, which uses maximum diagonal segments, is shown in the figure. For other details the figure is self- explanatory.



The minimum total elapsed time = Processing time for Job 1+ idle time for Job 1

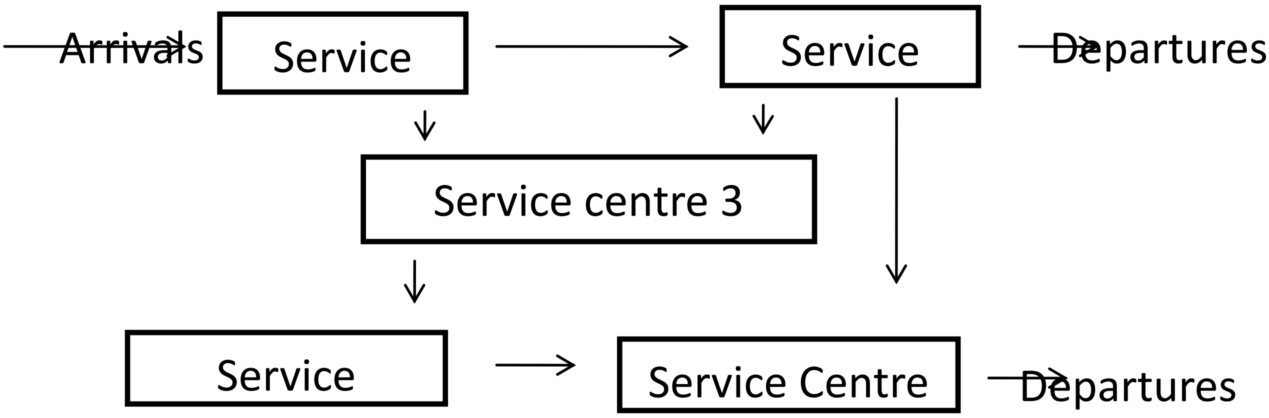
=12+3=15 hours (or)

The minimum total elapsed time = Processing time for Job 2+ idle time for Job 2

=12+3=15 hours

Queueing Theory

In every life, it is seen that a number of people arrive at a cinema ticket window. If the people arrive "too frequently" they will have to wait for getting their tickets or sometimes do without it. Under such circumstances the only alternative is to form a queue, called the waiting line, in order to maintain a proper discipline. Occasionally, it also happens that the person issuing tickets wil have to wait,(i.e, remains idle), until additional people arrive. Here the arriving people are called the customers and the person issuing the tickets is called a server.



Queuing theory is concerned with the statistical description of the behavior of queues with finding, e.g., the probability distributed of the number in the queue from which the mean and variance of queue length an probability distribution of waiting time for a customer, or the distribution of a server's busy periods can be found. In operational research problems involving queues, investigators must measure the existing system to make an objective assessment of its characteristics, and must determine how changes may be made to the system, what effects of various system's kinds of changes in the system's characteristics would be, and whether , in the light of the queuing system under study must be constructed in this kind of analysis and the results of queuing theory are required to obtain the characteristics of the model and to assess the effects, such as the addition of an extra server or a reduction in mean service time.

QUEUEING SYSTEM

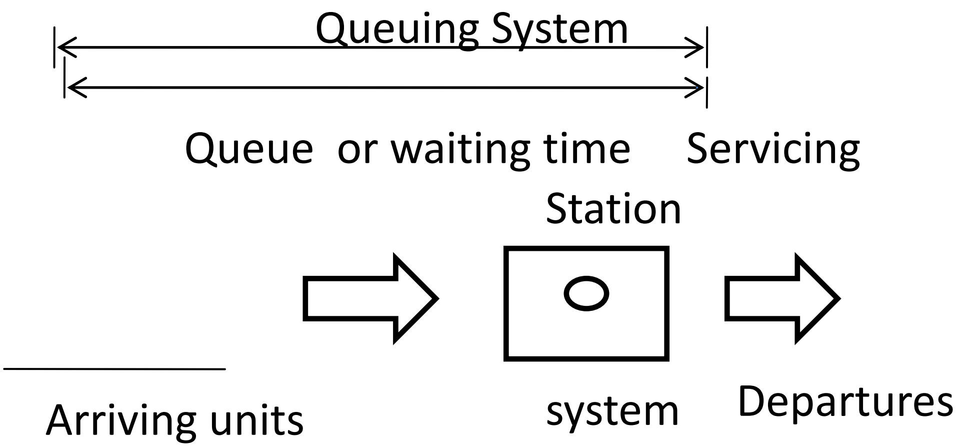
A queuing system can be completely described by

(a) The input (or arrival pattern),

(b) The service mechanism (or service pattern)

(c) The ' queue discipline ' and

(d) Customer's behavior.



(a) **The input (or arrival pattern)**. The input describes the way in which the customers arrive and join the system. Generally, the customers arrive in a more or less random fashion which is not worth making the prediction. Thus, the arrival patteren can best be described in terms of probabilities and consequently the probability distribution for inter – arrival times (the time between two successive arrivals) or the distribution of number customers arriving in unit time must be defined.

(b) **The service mechanism (or service pattern).** It is specified when it is known how many customers can be served at a time, what the statistical distribution of service time is, and when service is available. It is true in most situations that service time is a random variable with the same distribution for all arrivals, but cases occur where there are clearly two or more classes of customers, (e.g., machines waiting repair) , each with a different service time distribution. Service time which are important in practice are the ‘ negative exponential distribution’ and the related ‘ Erlang (gamma) distribution ‘. Queues with the negative exponential service time distribution are studied in the following sections.

(c) **The queue discipline.** The queue discipline is the rule determining the formation of the queue, the manner of the customer’s behavior while waiting, and the manner in which they are chosen for service. The simplest discipline is “ First come, first served”, according to which the customers are served in the order of their arrival. For example, such type of queue discipline is observed at a ration shop, at cinema ticket windows, at inrailway stations, etc., If the order is reversed we have the “ last come, first served “ discipline, as in the case of a big go down the items which come last or taken out first. An extremely difficult queue discipline to handle might be “ service in random order “ or “ might is right “.

Notations:

FIFO $\Rightarrow$ First In, First Out

FCFS $\Rightarrow$ First come, First Served

LIFO $\Rightarrow$ Last In, First Out

SIRO $\Rightarrow$Service in Random Order

FILO$\Rightarrow$ First In, Last Out.

(d) **Customer’s behavior.** The customers generally behave in four ways:

(i) **Balking .** A customer may leave the queue because the queue is too long and he has no time to wait, or there is not sufficient waiting space.

(ii) **Reneging .** This occurs when a waiting customer leaves the queue due to impatience.

(iii) **Priorities.** In certain applications some customers are served before others regardless of their order of arrival. There customers have priority over others.

(iv) **Jockeying.** Customers may jockey from one waiting line to another. It may be seen that this occurs in the supermarket.

Transient and steady states

Queuing theory analysis involves the study of a system's behavior over time. A system is said to be in " transient state " when its operating characteristics (behavior) are dependent on time . This usually occurs at the early stages of the operations of the system where its behavior is still dependent on the initial conditions. How ever, since we are mostly interested in the " long run " behavior been paid toward " steady state" .

A steady state conditions is said to prevail when the behavior of the system becomes independent of time. Let $P_n(t)$ denote the probability that there are n units in the system at time t. In fact, the change of $P_n(t)$ with respect to t is described by the derivative $\left[\frac{dP_n(t)}{dt}\right] or P_n{}'(t)$. Then the queuing system is said to become 'state' eventually, in the sense that the probability $P_n(t)$ is independent of time, that is, remains the same as time passes $(t \to \infty)$. Mathematically, in steady state

$$\lim_{n\to\infty} P_n(t) = P_n(independent of t)$$

$$\Rightarrow \lim_{t\to\infty}\left[\frac{dP_n(t)}{dt}\right] = \frac{dP_n}{dt}$$

$$\Rightarrow \log_{t\to\infty} P_n{}'(t) = 0.$$

In some situations, if the arrival rate of the system is larger than its service rate, a steady state cannot be reached regardless of the length of the elapsed time. In, fact, in this case the queue length will increase with time and theoretically it could bulid up to infinity. Such case is called the " explosive state".

In this chapter, only the steady state analysis will be considered. We shall not treat the ' transient' and ' explosive' states.

A list of symbols

Unless otherwise stated, the following symbols and terminology will be used hence forth in connection with the queueing models. The reader is reminded that a queueing that a queueing system is defined to include the queue and the service station both.

n=number of units in the system.

$P_n(t)$= transient state probability that exactly n calling units are in the queueing system at time t

E_n= the state in which there are n calling units in the system.

P_n= steady state probability of having n units in the system.

λ_n= mean arrival rate (expected number of arrivals per unit time) of customers (when n units are present in the system)

μ_n =mean service rate (expected number of customers served per unit time when there are n units in the system)

λ = mean arrival rate when λ_n is constant for all n

μ = mean service rate when μ_n is constant for all $n \geq 1$

s= number of parallel service stations.

$\rho = \frac{\lambda}{\mu s}$= traffic intensity (or utilization factors) for servers facility, that is, the expected fraction of time the servers are busy

$\varphi_{T(n)=}$ probability of n services in time T, given that servicing is going on throughout T.

Line length (or queue size)= number of customers in the queuing system.

Queue length= line length (or queue size) – (number of units being served)

$\varphi(w)$ = probability density function (p.d.f.) of waiting time in the system.

L_s =expected line length, i.e., expected number of customers in the system

L_q =expected queue length, i.e., expected number of customers in the queue

W_s = expected waiting time per customer in the system

W_q = expected waiting time per customer in the queue

(W|W>0) = expected waiting time of a customer who has to wait

(L|L>0)= expected length of non-empty queues. i.e., expected number of customers in the queue when there is a queue

P(W>0)= probability of a customer having to wait for service

$\binom{n}{r}$ = denotes the binomial coefficient $n_{C_r} = \frac{n!}{r!(n-r)!} = \frac{n(n-1)\ldots(n-r+1)}{r!}$ for r and n non- negative integers ($r \leq n$).

DISTRIBUTIONS IN QUEUEING SYSTEMS

In particular, we show that, if arrivals are ‘ completely random’, the number of arrivals in unit time has a Poisson distribution, and the intervals between successive arrivals are distributed negative exponentially.

Distribution of Arrivals ‘The Poisson Process’ (Pure Birth Process).

Arrival Distribution Theorem.

If the arrivals are completely random, then the probability distribution of number of arrivals in a fixed time – interval follows a Poisson distribution.

Proof: In order to derive the arrival distribution in queue, we make the following three assumptions

1. Assume that there are n units in the at time t, and the probability that exactly one arrival will occur during small time interval Δt be given by $\lambda\Delta t + o(\Delta t)$, where λ is the arrival rate independent of t and O(Δt) includes the terms of higher order of Δt.
2. Further assume that the time Δt is so small that the probability of more than one arrival in time Δt is $O(\Delta t)^2$, i.e., almost zero.
3. The number of arrivals in non- overlapping intervals are statistically independent, i.e., the process has independent increments.

We now wish to determine the probability of n arrivals in a time interval of length t, denoted by $P_{n(t)}$. Clearly, n will be an integer greater than or equal to zero. To do so, we shall first develop the differential equations governing the process in two different situations.

Case 1. When n >0

For n>0 ,there may be two mutually exclusive ways of having n units at time $t + \Delta t$

(i) There are n units in the system at time t and no arrival takes place during time interval Δt. Hence , there will be n units at time $t + \Delta t$ also.

Therefore, the probability of these two combined events will be = prob. Of n units at time t x prob. of no arrival during $\Delta t = P_{n(t)}.(1 - \lambda\Delta t)$

(ii) Alternately, there are (n-1) units in the system at time t, and one arrival takes place during Δt. Hence there will remain n units in the system at time t+Δt.

Therefore, the probability of these two combined events will be = prob.of (n-1) units at time t x prob.of one arrival in time Δt

=$P_{n-1}(t).\lambda.\Delta t$

We get the probability of n arrivals at time t+Δt

$$P_n(t + \Delta t) = P_n(t)(1 - \lambda\Delta t) + P_{n-1}(t)\lambda\Delta t$$

Case 2. When n=0

$$P_0(t + \Delta t) = P_0(t)(1 - \lambda\Delta t)$$

Dividing both sides by Δt and then taking limit as $\Delta t \to 0$

$$\lim_{\Delta t \to 0} \frac{P_n(t + \Delta t) - P_n(t)}{\Delta t} = -\lambda P_n(t) + P_{n-1}(t)$$

$$\lim_{\Delta t \to 0} \frac{P_0(t + \Delta t) - P_0(t)}{\Delta t} = -\lambda P_0(t)$$

$\mathrm{P}_{\mathrm{n}}^{'}(\mathrm{t}) = -\lambda P_n(t) + \lambda P_{n-1}(t)$ n>0

$\mathrm{P}_0^{'}(\mathrm{t}) = -\lambda P_0(t)$ n=0

$$\mathrm{P}(z, t) = \sum_{n=0}^{\infty} P_n(t) z^n$$

$$\mathrm{P}'(z, t) = \sum_{n=0}^{\infty} P_n{'}(t) z^n$$

Multiplying both sides of (A) by z^n and taking summation for n=1,2,…..,∞ we get

$\sum_{n=0}^{\infty} P_n{'}(t) z^n$ =-λ$\sum_{n=1}^{\infty} P_n(t) z^n$ + λ$\sum_{n=1}^{\infty} P_{n-1}(t) z^n$

$$\frac{P'(z,t)}{P(z,t)} = \lambda(z-1)$$

Log P(z,t)=λ(z-1)t+E

To determine E, we put t=0 to get

Log P(z,0)=$E^{'}$

P(z,0)=$\sum_{n=0}^{\infty} z^n \, P_n(0)$

=1+0

=1

P(z,t)=$e^{\lambda(z-1)t}$

Now , $P_n(t)$ can be defined as

$P_n(t) = \frac{1}{n!}\left[\frac{d^n \, P(z,t)}{dz^n}\right]$ t=0

In general, $P_n(t) = \frac{(\lambda t)^n e^{-\lambda t}}{n!}$

Classification of queueing models

I- Probabilistic queueing models

Model I. (Erlang Model). This model is symbolically represented by (M|M|I):($\infty|FCFS$). this denotes poisson arrival (exponential inter -arrival) poisson departure (exponential service time) single server , infinite capacity and " first come , first served " service discipline.

Model II. (General Erlang Model) . Although this model is also represented by (M|M|I):($\infty|FCFS$) , but this is a general queueing model in which the rate of arrival and service depend on the length n of the line.

Model III this model represented by (M|M|I):($N|FCFS$). In this model capacity of the system is limited (finite) , say n. Obviously , the number of arrivals will not exceed the number N in any case .

Model IV. This model is represented by (M|M|S):($\infty|FCFS$) , in which the number of stations is s in parallel .

Model V. This model is represented by (M|E_k|I):($\infty|FCFS$), that is , poisson arrivals , Erlangian service time for k phases and a single server.

Model VI. (Machine servicing model) .this model is represented by (M|M|R):$(K|GD), k > R$, that is , poission arrivals , exponential service time , R repairmen , and K machines in the system and general queue discipline.

Model VII. Power -supply model. Models.

Model VIII. Economic cost profit models.

Model IX. (M|G|I):$(\infty|GD)$, where G is the general output distribution , and GD represents a general service discipline

II- Mixed Queueing Model

Model X. (M|D|I):$(\infty|FCFS)$, where D stands for deterministic service time.

III- Deterministic Queueing Model

Model XI. (D|D|I):$(K-1|FCFS)$, where
D→ Deterministic arrivals , i.e., interarrival time distribution is constant or regular.
D→ Deterministic service time distribution.

Solution of queueing models

(a) To obtain system of steady state equations goverening the queue
(b) To solve this equations for finding out the probability distribution of queue length.
(c) To obtain probability density function for waiting time distribution
(d) To find the busy period distribution
(e) To derive formula for $L_s, L_q, (L|L>0), W_s, W_q, (W|W>0)$, and var (n) etc.,
(f) Also , to obtain the probability of arrival during the service time of any Customer

To solve the system of difference equations.

To solve difference equations:

0=-(λ+$\mu)P_n + \lambda P_{n+1} + \mu P_{n+1}$ $if\ n > 0$

0=-λ$P_o + \mu P_{1,}$ if n=0

We use the technique of successive substituation

Since $P_0 = P_0$

$$P_1 = \frac{\lambda}{\mu} P_0$$

$$P_2 = \frac{\lambda}{\mu} P_1$$

.

.

.

.

$$P_n = \frac{\lambda}{\mu} P_{n-1}$$

Now using the fact that $\sum_{n=0}^{\infty} P_n = 1$

$P_0[1 + \frac{\lambda}{\mu} + (\frac{\lambda}{\mu})^2 \ldots\ldots]$=1

$$P_0 \left[\frac{1}{1 - \lambda/\mu} \right] = 1$$

=1-p

Now substituting the value of P_0 from

$P_n = (\frac{\lambda}{\mu})^2$(1-$\frac{\lambda}{\mu}$)

=ρ^n(1-ρ)

Measure of Model I:

To find expected number of units in the system L_s

By definition of expected value,

$$L_s = \sum_{n=1}^{\infty} nP_n$$

$=\sum_{n=1}^{\infty} n(\frac{\lambda}{\mu})^n\ (1-\frac{\lambda}{\mu})$

$= (1-\frac{\lambda}{\mu})\frac{\lambda}{\mu}\sum_{n=1}^{\infty} n(\frac{\lambda}{\mu})^n - 1$

$=(1-\frac{\lambda}{\mu})\frac{\lambda}{\mu}[\frac{1}{1-\lambda/\mu}]$

$$L_s = \frac{\lambda/\mu}{(1-\lambda/\mu)} = \frac{\rho}{1-\rho}, where\ \rho = \frac{\lambda}{\mu} < 1)$$

Which is the required formula

To find expected queue length L_q

Since there are (n-1) units in the queue excluding one being serviced,

$$L_q = \sum_{n=1}^{\infty}(n-1)P_n$$

$= \sum_{n=1}^{\infty} nP_n$ -$\sum_{n=1}^{\infty} P_n$

$=\sum_{n=1}^{\infty} nP_n$ -$[\sum_{n=1}^{\infty} nP_n - P_0]$

$=L_s = [1 - P_0]$

Substituting the value of P_0

$L_q = L_s - 1 + (1 - \frac{\lambda}{\mu})$

$L_q = L_s - \frac{\lambda}{\mu}=\frac{\rho^2}{1-\rho}$ where $L_s = \frac{\lambda/\mu}{(1-\lambda/\mu)} = \frac{\rho}{1-\rho}$

To find expected waiting time in the system W_s

Since expected waiting time in the system

= Expected waiting time in queue + expected time, $W_s = W_q + 1/\mu$

$$W_s = \frac{\lambda}{\mu(\mu-\lambda)} + \frac{1}{\mu} = \frac{1}{\mu-\lambda}$$

To find variance of queue length

By definition

Var. {n}=$\sum_{n=1}^{\infty} n^2 P_n - [\sum_{n=1}^{\infty} nP_n]^2$

=$\sum_{n=1}^{\infty} n^2 P_n$-$[L_s]^2$

=$\sum_{n=1}^{\infty} n^2 (1-\rho)\rho^n - [\frac{\rho}{1-\rho}]^2$

=${}^{\rho}/_{(1-\rho)^2}$

Inter- Relationship Between L_s, L_q, W_s, W_q

It can be proved under general conditions of arrival, departure, and discipline that the formulae

$$L_s = \lambda W_{s,}$$

And $L_q = \lambda W_q$

Will hold in general. These formulae act As key poits in establishing the strong relationship between L_s, L_q, W_s, W_q which can be found as follows

By definition,

$$W_q = W_s - {}^{1}/_{\mu}$$

Thus, multiplying both sides by and substituting the values

$$L_q = L_s - {}^{\lambda}/_{\mu}$$

This means that one of the four expected values should immediately yield the remaining three values.

MODEL IV (A) : (M|M|S): ($\infty|FCFS$)

In this more realistic queueing system the customers arrive in a poission fashion with mean arrival λ. There are s number of counters arranged in parallel, and a customer can go tc any of the free counters for his service, where the service time at each counter is identical and follows the same exponentioal distribution law. The mean service rate per busy service is μ. Therefore over all service rate, when there are n units in the system may be obtained in the following two situations.

(i) If $n \leq s,$ all the customers may be served simulataneously. There will be no queue , (s-n) number of servers may remain idle, and then

$$\mu_n = n\mu, n = 0,1,2, \dots., s;$$

(ii) If $n \geq s$, all the servers are busy , maximum number of customers waiting in queue will be (n-s), then $\mu_n = n\mu$

$MODEL$ IV (B) : (M|M|S): $(N|FCFS)$

In model IV(A) , if the maximum number in the system is limited to N, then as in model III,

$$\lambda_n = \begin{cases} \lambda \, for \, 0 \leq n \leq N \\ 0 \quad for \, n > N \end{cases}$$

And

$$\mu_n = \begin{cases} n\mu \, for \, 0 \leq n \leq s \\ s\mu \quad for \, s \leq \, n \leq N \end{cases}$$

Virtually the same relationships hold between P_n and P_0 as in model IV(A) with infinite capacity. Therefore,

$$P_0 = \begin{cases} (s\rho)^n \, {P_0}/{n!} \;\; for \, o \leq n \leq s \\ s^n \rho^n \, {P_0}/{s!} \qquad for \, s \leq n \leq N \\ 0 \, for \, n > N \end{cases}$$

Where P_0 may be written as

$$P_0 = [\sum_{n=1}^{s-1} \frac{(sp)^n}{n!} + \sum_{n=s}^{N} {(sp)^n}/{s!} \, (s^n - s)]^{-1}$$

$$= \begin{cases} [\sum_{n=0}^{s-1} \frac{(sp)^n}{n!} + \frac{(s\rho)^s}{s!(1-\rho)} (1 - \rho^{N-s+1})]^{-1} \quad , \rho = \frac{\lambda}{s\mu} \neq 1 \\ [\sum_{n=0}^{s-1} \frac{(sp)^n}{n!} + \frac{(s\rho)^s}{s!} (N - s + 1)]^{-1} \quad , \rho = \frac{\lambda}{s\mu} = 1 \end{cases}$$

This queueing modrl with limited waiting room is valuable because of its relevance to many real situations and the fact that changes may be made to its properties by adjusting the number of servers or the capacity of the waiting room. However, while poission arrivals are common in practice, negative exponential service times are less so , and it is the second assumption in the system M|M|s that limits its usefulness.

Example;

A supermarket has two girls ringing up stales at the counters. If the service time for each customer is exponential with mean 4 minutes and if people arrive in a poission fashion at the counter at the rate of 10 per hour, then calculate.

(a) The probability of having to wait for service;

(b) The expected percentage of idle time for each girl;

(c) If a customer has to wait, find the expected length of his waiting time.

Solution

(a) Probability of having to wait for service is

P(W>0)=$\frac{({}^{\lambda}/_{\mu})^{s}}{s!(1-\rho)}P_0$

λ=1/6,$\mu=\frac{1}{4}, s=2$

$$\rho = {}^{\lambda}/_{\mu s} = \frac{1}{3}$$

Now, compute

$$P_0 = [\sum_{n=0}^{s-1}\frac{(sp)^n}{n!} + \frac{(s\rho)^s}{s!\,(1-\rho)}]^{-1}$$

$$= \sum_{n=0}^{1}[\frac{(2.{}^{1}/_{3})^n}{n!} + \frac{(2.{}^{1}/_{3})^2}{2!\,(1-{}^{1}/_{3})}]^{-1}$$

=[1+2/3+4/9/2x2/3]$^{-1}$=1/2

Thusprob.(W>0)=$\frac{({}^{4}/_{6})^{2}.0.5}{2!(1-{}^{1}/_{3})} = {}^{1}/_{6}$

(b) The fraction of the time the service remains busy is given by $\rho = {}^{\lambda}/_{\mu s}$ =1/3

Theefore, the fraction of the time thee service remains idle is= (1-1/3)=2/3=67%(nearly)

(c) (W|W>0) =$\frac{1}{1-\rho}.\frac{1}{s\mu} = \frac{1}{1-{}^{1}/_{3}}.\frac{1}{2\times{}^{1}/_{4}} = 3\,minutes$

****

UNIT- IV

Inventory Control

Introduction

A inventory may be defined as an idle resources that possesses economic value. It is an item stored or reserved for meeting future demand. Such items may be materials machines, many or even human resources.

Reasons for holding storks

The main reasons are

(1) To ensure sufficient goods one are available to meet anticipated demands.
(2) To absorb variations in demand and production.
(3) To prove a buffer between production processor.
(4) To take advantage of bulk purchasing discounts.
(5) To meet possible shortages in the future.
(6) To enable produces process to flow smoothly and efficiently
(7) As deliberate investment policy particularly in times of inflation or possible shortages.

The Objective of Inventory Control

The objective of inventory control is to maintain stock levels so that the combined costs, mentioned earlier are at amenities. This is done by establishing two sectors.

“how to order? When to order?”Inventory control terminologies.

(1) Demand : The amount of quantity required by sales or products usually expressed as the rate of demand for week or month or year etc.

(2) Economic order quantity (EOQ)
This is a calculated ordering quantity which minimized the balance between inventory holding costs and re-order costs.
(3) Lead time: The period of time between ordering and replenishment.
(4) Butter stock (or safety stock). It is a stock allowance to cover errors in forecasting the lead time on the demand during the lead time.

(5) Maximum stock : A stock level as the maximum desirable which is used as an indicator.

(6) Reorder level: The level of stock of which culture replacement order should be placed. The re-order level 9 independent upon the lead time and the demand upon the lead time and the demand during the lead time .

Types of Inventory

Models:

There are two types of models can be used for inventory control.

They are (i) deterministic model (ii) stochastic model.

A deterministic model is one which assumes complete continuity. The values of all factors like demand, usage, lead time, carrying costs etc are known exactly and there is no element of risks and uncertainty.

Stochastic model exists where some of all the factors are not known with costmity and only they are expressed with probabilities.

Model: (Purchasing model method shortage)

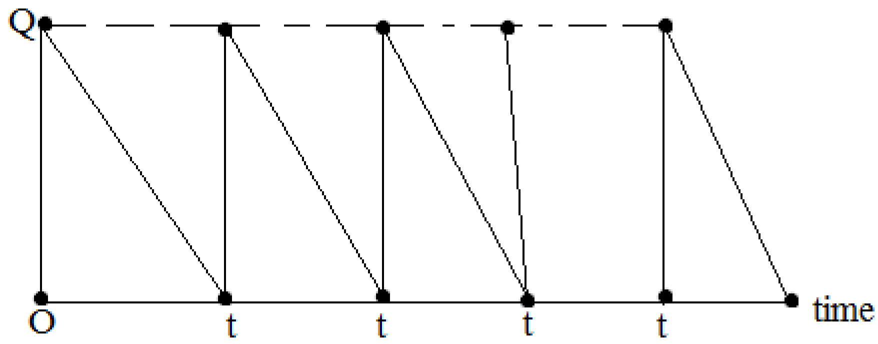
In this model, the following assumptories, are made

1) The annual demand, D is uniform and is known.
2) The replenishment is made instantaneity.
   (ie) the whole batch is delivered at once.
3) There is no shortages and no lead time.
4) There is known constant inventory carrying cent (C_1).
5) There is known constant ordering cost (C_3).
6) There is known constant unit prince.

Let Q be the ordering quantity per order.

Let of be the time interval between

Inventory level



The total cost per year

= Total ordering cost + total – inventory carrying cost.

$C_4(Q) = G(Cs + \frac{1}{2} Q\, C_1)$

$= \frac{D}{Q}cs + \frac{1}{2}C_1$

$(\therefore D = GQ)$

Differenhating w .r. to Q

We get

$\frac{d}{dQ}(C_A\, \phi) = \frac{D}{Q^2}(s + \frac{1}{2} C_1$

$\frac{d^2C_A(Q)}{dQ^2} = \frac{2D}{Q^3}$ G is always posture

$\frac{d}{dQ} = 0$ is given by $Q^2 = \frac{2DG}{C_1}$

$\therefore Q = \sqrt{\frac{2DG}{C_1}}$

For this value of Q, $\frac{d^2}{dQ^2} C_A > 0$

“ C_A (Q) in minimum

When $Q = \sqrt{\frac{2DG}{G}}$

This ordering quantity is denoted by Q.

$\therefore$.toQ us $Q^o = \sqrt{\frac{2DG}{C_1}}$

Substituting this in (1) we get the total minimum cost as

$$CA\ (Q^o) = DG\sqrt{\frac{2DG}{2DC_s}} + \frac{1}{2}\ C_1\sqrt{\frac{2DG}{C_1}}$$

$$= \frac{1}{2}\sqrt{2\mathrm{D}C_1}Cs + \frac{1}{2}\sqrt{2\mathrm{D}C_1}Cs$$

$$= \sqrt{2\mathrm{D}C_1}Cs$$

The optimum time interval between order is given by $t^o = \frac{Q^o}{D}$.

The optimum number of orders in a year is n $t^o = \frac{Q^o}{D}$.

Note:-

(i) Total minimum cent is also by $C_4(Q^o) = Q^o{*}C_1$,

(ii) For calculating the total minimum cost we consider only inventory carrying cost. The cost of material is always a constant

Worked Example

1. Annual demand for an item is 3200 units. The unit cent is Rs.6 and inventory carrying changes 25% per annum. If the cost of one procurement is Rs.150, determine.

(i) Economic order quantity.
(ii) Number of orders per year.
(iii) Time between the two consecutive order.
(iv) The optimal cost.
(v) The optimal cost

Solution:

Annual Demand (D) = 3200units.

Procurement cost (C_s) = Rs. 150

Inventory carrying cost (C_1) = Rs. 6x25=100, Rs.150

(i) EOQ is $Q^o = \sqrt{\frac{2DCs}{C_1}} = \sqrt{\frac{2x3200x150}{150}}$ = 800 units

(ii) Number of orders per year (u)

$= \frac{D}{Q^o} = \frac{3200}{800} = 4$

(iii) Optical time between two consecutive orders to = $\frac{Q^o}{D} = \frac{800}{3200} = \frac{1}{4}$yr or 3months.

(iv) The optical cost $\sqrt{2x3200x1.50x150}$

= Rs. 1200

2. A company buys its annual relativements of 36,000 unitsin six instalments. Each unit costs Rs.1 and ordering cost is Rs.25. The inventory carrying cost is estimated at 20% of unit value Find the total annual cost of the existing inventory policy. How much money can be saved by using EOQ.

Solution:-

D = 36000 units per year

Number of orders = 6

Ordering quantity per order (ϕ) $\frac{36000}{5}$

Inventory carrying cost = 1x0.20 = Rs. 0.20

Ordering cost (C_s) = Rs. 25

Total Cost as per the existing policy

= Total ordering cost + total carrying cost

= n x C_s + $\frac{1}{2}$ = QxC_1

= 6x25+$\frac{1}{2}$x6000x20

=150+600=Rs. 750

EOQ= $\sqrt{\frac{2DCs}{C_1}} = \sqrt{\frac{2x3200x150}{0.20}}$

= 3000 units

Total minimum cost $\sqrt{2DGCs}$

= $\sqrt{2x3600x0.20x25}$ = Rs. 600

Amount of savings by following EOQ policy = Rs.750-Rs.600=Rs.150

Note: Total minimum cost = $\phi^{o}xC_1$

= 3000x02=600 Rs.

Total ordering cost = $\frac{36000}{3000}$x 25

= Rs.300

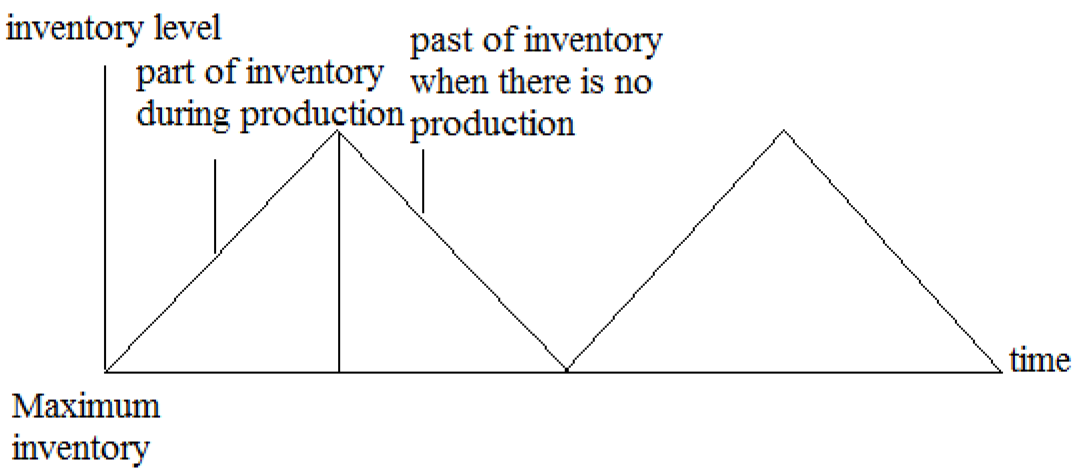
Model II

Production model without shortages

(Manufacturing model – No Shortaged

In this model we assume that the ordering quaintly is received only over a period of time. Further is assumed that the production rate is greater than the demand rate, otherwise there will be no inventory build up and stock only will occur. These two assumptions are made in addition to the assumption of model I. Let k be the production rate, r be the demand rate, otherwise there will be no inventory build up and stock only will occur. There two assumptions are made in addition to the consumption of model I.

Let k be the production rate, r be the demand rate. Let tbe the length of production run rate, r be the demand rate. Let t be the length of production run



Total production = Q = kt t= $\frac{Q}{K}$

The average height is half the height of the triangle and the height in determined by the rate of replenishment less the demand over the replimshment.

period= $\frac{kt}{2}$ - $\frac{rt}{2}$

$= \left(\frac{k-r}{k}\right)G$

Total cost = total Ordering cost + Total Carrying cost

$= u\ C_s + \frac{Q}{2}\ (1-\frac{r}{K})\ G$

$CA(Q) = \frac{r}{\phi}cs + \frac{Q}{2}\ (1-\frac{r}{K})G$

$\frac{d}{da}\ CA = \frac{2rCs}{Q^2} > 0$ always

$\frac{d}{dQ}\ CA = O \Rightarrow Q^2 = 2rCs\ \left(\frac{r}{K-r}\right)$

Total minimum cost is given by

$$CA\ (Q^o) = rcs\sqrt{\frac{k-r}{2rGK}} + \frac{1}{2}\frac{k-r}{2rGK}\ G\ \sqrt{\frac{2Gr\,(k-r)}{GK}}$$

$$= \sqrt{\frac{2Gr\,(k-r)}{GK}}$$

The optimum time interval between orders is $t^o \frac{Q^o}{r}$

The optimum of number of order is

$$n = \frac{r}{\phi^o}$$

Worked Example

A manufacturer has to supply his customer with 600 units of his product per year. Shortages are not allowed and storage cost amounts to 60paise per unit per unit per year. The set up cost associated with ordering 200% higher than EOQ.

Solution

D = 600 units per year

Cs= Rs.80 per setup

C_1 = Rs. 0.60

(i) EOQ = $\sqrt{\frac{2DG}{G}} = \sqrt{\frac{2x600x80}{0.60}}$ = 4000 units.

(ii) The minimum average yearly cost
= $Q^o x C_1$ = 4000x0.6 = Rs. 240

(iii) The optimum number of per year

$\frac{D}{\phi^o} = \frac{600}{400}$ = 1.5

(iv) The optimum period of supply = $\frac{\phi^o}{D} = \frac{400}{600} = \frac{2}{3}$ year = 8 months.

(v) 200% of EOQ = 400 x $\frac{20}{100}$ = 80

∴ New ordering quality = 480 units

T.C. nCs + $\frac{1}{2}$ Q C_1

= $\frac{600}{480}$ x 80 + $\frac{1}{2}$ x 480 x 0.60

Increase is cent

= Rs. 244 – Rs. 240

= Rs. 4

Model III [Purchasing model with shortage]

Assumptions

(1) Annual demand 'D' is uniform at a rate of r units per unit time.
(2) Production is instantaneous.
(3) Zero lead time
(4) G is the inventory carrying cost per units per year.
(5) C_s is the setup cost
(6) C_2 is the shortage cost per unit/yr

$$Q = \sqrt{\frac{2DCs}{C_1}\left(\frac{C_1+C_2}{C_2}\right)}$$

Maximum inventory level = x $\left(\frac{C_2}{C_1+C_2}\right)$ and t^o = $\frac{\phi}{D}$

Minimum cost CA (ϕ) = $\sqrt{2DC_1Cs\ \frac{C_2}{C_1+C_2}}$

Worked Example

Consider the following data

Unit cost = Rs. 100

Order Cost = Rs. 160, Inventory carrying

Cost = Rs. 20, Back order (stock out cost) Rs. 10

Annual demand = 1000 unit

Computed the following

(i) Minimum order quantity
(ii) Time between
(iii) Maximum number of backorder
(iv) Maximum inventory level
(v) Overall annual cost

Solution:

D = 1000 units, Cs=Rs.160, C_1=Rs.20 C_2= Rs.10.

(i) Minimum cost order quantity = $\sqrt{\frac{2x1000x160}{10}}$

(ii) = 219 units.

(iii) Time between orders: f* = $\frac{\phi*}{D}$ = $\frac{219}{1000}$x12

= 2.628 months

Maximum number of backorders

=ϕ*-ϕ* = 219-73 = 146 units

(iv) Maximum Inventory level

Q_1* = Q* $\left(\frac{G}{C_1+C_2}\right)$ = 219 $\left(\frac{10}{20+10}\right)$ = 73 units

v) Over all annual cent = 1000x100 $\sqrt{2DC_1Cs\left(\frac{G}{C_1+C_2}\right)}$

$= 100000+ \sqrt{2x1000x20x160x\frac{10}{30}}$ = Rs.

1,01,306

Model IV [Manufacturing model with shortage]

1. Demand D is uniform at a rate of 'runits' per unit time.
2. Production rate is finite say k>r units per unit time
3. Shortages are allowed and have logged
4. Zero lead time
5. C_1= inventory carrying cost per unit time.
6. C_2 = shortage cost per unit line
7. Cs = set up cost per run.

Optimal order quantity,

$$Q = \sqrt{\frac{2csr\,(C_1+C_2)}{C_1C_2\left(1-\frac{r}{k}\right)}}$$

Minimum cost Q* = $\sqrt{\frac{2csr\,(C_1+C_2)}{C_1C_2\left(1-\frac{r}{k}\right)}}$

$S^* = \frac{Q^*C_1(1-\frac{r}{k})}{C_1+C_2}$

$t^*=\frac{Q*}{r}$, minimum cost in

$$C^* = \sqrt{\frac{2r\ C_1C_2Cs\ (1-\frac{r}{Q})}{C_1+C_2}}$$

Worked Example:-

1. An item is produced at the rate of 50 items per day. The demand occuss at the rate of 25 items per day. If the setup cost is Rs.100 per set up and holding cost is Rs. 0.01 per unit of item per day. Find the economics lot size for one run, assumes that the shortages are not allowed. Also find the time fo cycle and the minimum total cost of one run.

Solution

k = 50 perday

r =25 per days

Cs = Rs. 100

C_1 = Rs. 0.01

$Q^o = \sqrt{\frac{2Csr}{C_1}\left(\frac{k}{k-r}\right)}$

$Q^* = \sqrt{\frac{2x100x25}{0.01}\left(\frac{50}{50-25}\right)}$ = 100 units

$t^* = \frac{Q*}{r} = \frac{1000}{25}$ = 40 days

CA (Q*) = $\sqrt{24GCs\left(\frac{k}{k-r}\right)}$

= $\sqrt{2x25x0.01x100\left(\frac{50-25}{50}\right)}$ = Rs.5

Total variable cost per run = 5x40 = Rs. 200

Model V- Price Breaks Model we consider the case that the unit price vary with ordering quantity usually some discoints in unit price are offered to encourage having large purchasing quantity. Above the rates of discoint may vary depending on the purchasing quantity. These types of inventory models or prove break models. The general pattern of price breaks is given in the following table showing the ordering quantity and the price breaks.

Ordering quantity	unit price
$0\leq\{_1<b_1$	P_1
$b_1\leq\{_2<b_2$	P_2
$b_2\leq\{_3<b_3$	P_3
- - - - - - - - - - - -	- - - - -
$B_{n-1}\leq\{_n<b_n$	P_n

Hence $P_1 > P_2 \ldots\ldots > P_n$. The price of item falls as the ordering quantity is $b_1 b_2., b_{n-1}$ and so these ordering quantity level are called price breaks consider the inventory model with one price break.

Ordering quantity	unit price
$0 \leq Q_1 < b$	P_1
$b_1 \leq Q_2 < b_2$	P_2
$b_2 \leq Q3$	P_3

Working rule

1. Calculate Q_3* taking the lead unit price P_3. If $Q_3^* > b_2$ then Q_3* is the EOQ.
2. If $Q_3^* < b_2$ there calculate Q_2*. Now compare units $C(Q_2^*)$ and b_2 If $C(Q_2^*)$ in the EOQ. If (b_2) is the minimum then b_2 is the EOQ.
3. If $Q_2^* < b_1$ then, calculate Q_1* New compare the costs $(Q_1^*))$ and (b_1) and $((b_2)$

   If C(Q*) is minimum EOQ is Q_1*
   If $C(b_1)$ is minimum EOQ is b_1
   If $C(b_2)$ is minimum EOQ is b_2

Example:

Find the optimal order quantity for a product for which the prince leaks are us follows.

Quantity	Purchasing cost
$0 \leq \phi_1 < 100$	Rs. 20/ unit
$100 \leq \phi_2 < 200$	Rs. 18/unit
$200 \leq Qs$	Rs.14/unit

The marthly demand for the provided is 400 units. The storage cost is Rs. 20% of the unit cent of the product/month and cost of undering is Rs.25/-

Solution

D = 400 units permonth

Cs = Rs.25

C_1 = 20% of unit per month

Q_3* does not lie in the last range

∴.Q_3* can not be the EOQ.

$Q_3^* = \sqrt{\frac{2x400x25}{0.20x18}}$ = 75 units

This does not lie in the range

$100 \le Q_2 < 200$

$Q_1^* = \sqrt{\frac{2x20x25}{0.20x20}}$ = 75 units

Now we have to compare

CA(Q_1*) , CA(100), CA(200) to find EOQ.

CA (Q_1*) = cost of the material +total inventory cost

= 400x20+$\sqrt{2DC_1C_2}$

= 8000+$\sqrt{2x400x4x25}$

= 8283 nearly

$C_A(100)$ = 400x18 + $\frac{400}{100}$x25 $\frac{1}{2}$ x 100 x 3.6

= 7200 + 100+180

= Rs. 7480

$C_A(200)$ = 400 x 16 + $\frac{400}{100}$x25x $\frac{1}{2}$x200x3.2 = Rs. 6770.

Since total, cost is minimum for 200 units, EOϕ = 200 units.

Models : Probabilistic model (stochastic model)

In this model, we have to determine the optimum inventory evel which mentioned the expected cost. The following assumptions are made.

(i) T is constant internalbetween orders.
(ii) Z is the stock level at the beginning of each period of time ‘t’
(iii) Lead time is zero.
(iv) C_1 is the holding cent per unit time.
(v) C_2 is the shortage cost per unit time.
(vi) Also assume that the demand is instantionous ‘r’ be the demand, p(r) be the probability that demand is r units in time ‘t’.

Two cases: discrete of continuous

Discrete case: 4 is discrete If $\sum_{r=0}^{z-1} p(r) < \frac{C_2}{C_1+C_2} < \sum_{r=0}^{Z} p(r)$ then C(z+1) = C(z) and the optimum stock level is either Zo or Z_o+1

Newspaper boy problem

When r and z are discrete values, the discrete stochastic model is hand as news-paper boy problem. The news boy buys certain number of papers Z and sell some or all of them. He make profit for each sold paper and less for each unsold paper. All of demand are than the stock on anyday the ne is in the problem to find the optimum number of newspaper he was us buy each day so as to maximize his profit.

ABC Analysis

The university of an organization may consist of several items varying with cents usage, lead time, procurements cents and so on. Here we should pay more attention on claims with low usage value. So we should be selective to our approach towards inventory central. This is called relative inventory control and hence we have ABC analysis (always better control analysis).

Example:

The items kept in inventory by the school management studies at state university are instead below which claims should be classified as 'A' claims, 'B' less 'c' claims? What percentage of claims in each land.

Item	Annual usages	Value per unit
1	200	400.00
2.	100	366.00
3.	2000	0.20
4.	400	20.00
5.	6000	20.00
6.	1200	0.04
7.	120	0.80
8.	2000	100.00
9.	1000	0.70
10.	80	400.00

Solution:

In order to classify the items kept in the A,B,C Class items the first step into calculate the annual usuage value in the descending order.

S.No	Annual Usage	Value per item	Annual usage	Ranking Rs.

1	200	40	8000	IV
2	100	360.00	36000	I
3	2000	0.20	400	IX
4	400	20.00	8000	V
5	8000	0.04	240	X
6	1200	0.80	960	VIII
7	120	100.00	1200	III
8	2000	0.70	1400	VI
9	1000	1.00	1000	VII
10	80	400.00	32000	II

The table given below classifies the above items under three class A,B,C

Item	**Rank**	**Annual Usage Value**	**36000 32000**	**Class**	**Annual Usage Value ingots**	**% of total items**
2 10	I II	36000 32000	A	68000	$\frac{2}{10}$=206	$\frac{68000x10}{10000}$=68%
7 4 1	III IV V	12000 8000 8000	B	28000	$\frac{3}{10}$=30%	$\frac{68000x10}{100000}$=68%
8 9 6 3 5	VI VII VIII IX X	1400 1000 960 400 240	C	4000	$\frac{5}{10}$=50%	$\frac{4000x10}{100000}$=68 = 100

Iron the table it is clear that 20% of inventory claims belonging to A class, controller 68% of total annual usage value 30% falls under class B, 50% falls in C.

****

UNIT-V

Network Analysis

Network Analysis is a technique which determines the various sequences of jobs concerning a project and the project completion time. Network analysis has been successfully used to a wide range of significant management problems. Two popular methods of this technique which are widely used are 1) The Critical Path Method (CPM) and 2) The Programme Evaluation and Review Technique(PERT).

CPM: This method differentiates between planning and scheduling. Planning refers to the determination of activity and the order in which such activities should be performed to achieve the objective of the project. Scheduling refers to introduction of time in performing the jobs concerning the project taken in the plan. CPM technique is generally applied to well known projects where the time schedule to perform the activities can exactly be determined. In CPM two types of times are used. They are i) Normal time(time taken to perform an activity), ii) Crash time(the shortest time in which an activity can be finished)

PERT: This was specially developed for planning and controlling the polaris missile programme. Using this technique the management was able to obtain the expected project completion time and the bottle neck activities in a project. In PERT three types of times are used. They are i) Optimistic time estimate, ii) Pessimistic time estimate, iii) Most likely time estimate. In PERT, statistical analysis is used in determining the time estimates.

A path of network is the sequence of activities starting from the initial events to the final event proceeding in the direction of arrows. The duration of a path is the sum of activities coming along the path.

The path that takes the longest duration is called the critical path. The duration of this path is called the project duration. We introduce the following definitions.

(1) EST: It is the earlier possible time at which an activity can be started assuming that all the preceding activities can be started at the earliest start times.
(2) EFT: The earliest finish time for an activity is the earliest start time plus the duration of the activity. Ie)EFT=EST+Duration.
(3) LFT: The latest finish time for an activity is the latest possible time at which an activity can be finished assuming that all the subsequent activities can also be finished at the latest possible times.
(4) LST: The largest start time of an activity is equal to the latest finish time minus the duration of the activity.
(5) Total Float(TF): TF= Head event L-Tail event E-Duration

TF=LFT-EFT=LST-EST.

(6) Free Float(FF): FF=Head event L-Tail event E-Duration

FF=TF-Head event Slack(L-E)

(7) Independent Float(IF): IF=Head event L-Tail event E-Duration

=FF-Tail event slack(L-E).

Dummy activity: This is the activity which doesnot consume time or resources. It is used only to show logical dependencies between activities so as not to violate the rules for drawing networks. It is represented by a dotted arrow line in a network diagram.

PERT Network: A PERT network is drawn in the same way as a CPM network. In PERT network we need only to estimate the duration of each activity. There are 3 types of time estimates.

1. The Optimistic time estimate(T_0): It is the shortest possible time estimate for finishing an activity. The chance of occurrence of this is very small.
2. The Pessimistic time estimate(T_p): It is the longest time conceivable for an activity. The chance of this occurrence is also very small.
3. The most likely time estimate(T_m): This is the the time estimate to be executed under normal conditions of the activity. This is a reasonable time to estimate.

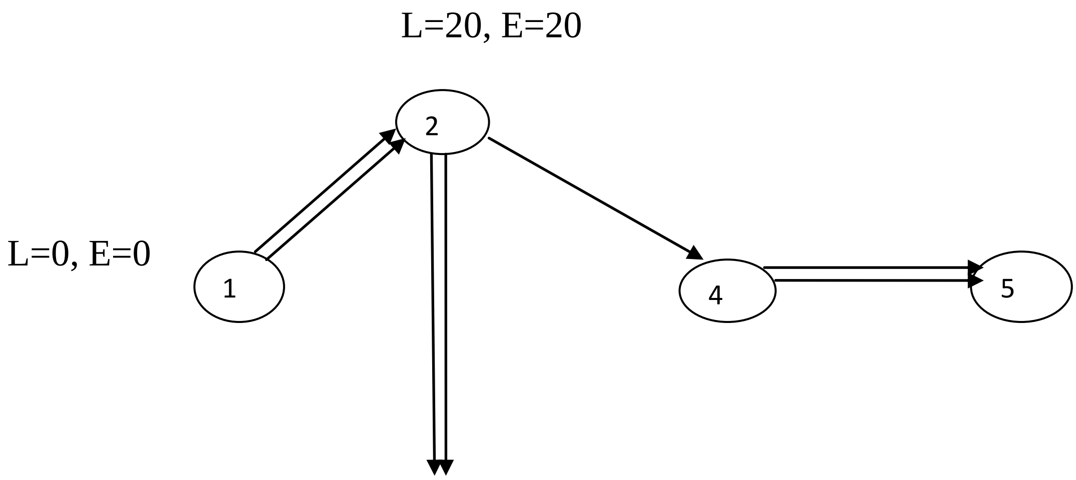
Problem 1:

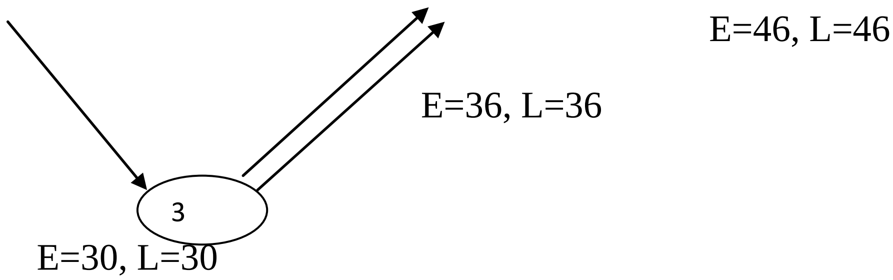
The following table gives the activities in a construction project and other relevant information.

Activity	1-2	1-3	2-3	2-4	3-4	4-5
Duration(Days)	20	25	10	12	6	10

(i) Draw the network for the project.
(ii) Find the critical path and the project duration.
(iii) Find the total float for each activity.

Solution:





Activity	Duration	EST	EFT	LST	LFT	TF
1-2	20	0	20	0	20	0
1-3	25	0	25	5	30	5
2-3	10	20	30	20	30	0
2-4	12	20	32	24	36	4
3-4	6	30	36	30	36	0
4-5	10	36	46	36	46	0

The critical path is 1-2-3-4-5

The project duration is 46 days.

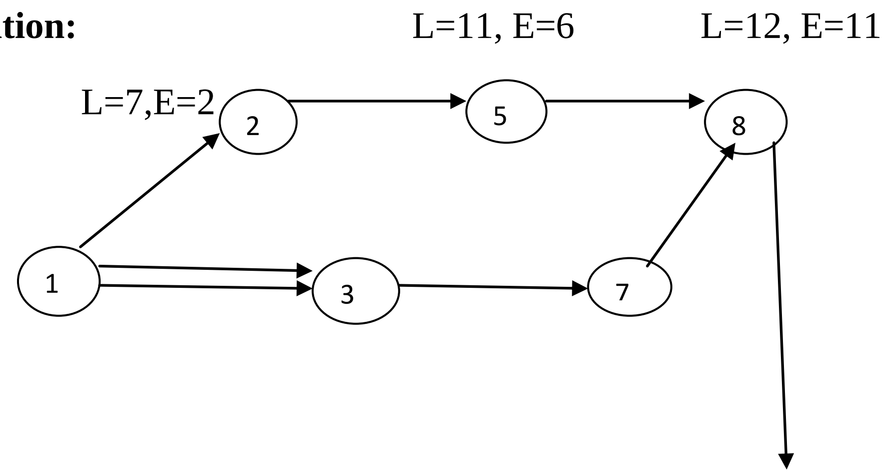
Problem 2:

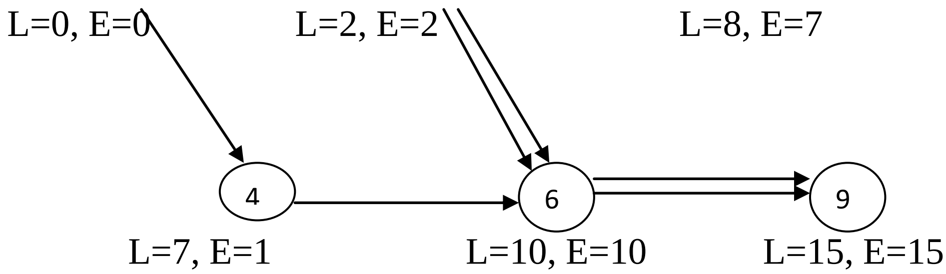
A project has the following time schedule.

Activity	Time in months	Activity	Time in months
1-2	2	3-7	5
1-3	2	4-6	3
1-4	1	5-8	1
2-5	4	6-9	5
3-6	8	7-8	4
		8-9	3

(i) Construct the network
(ii) Find the total float for each activity
(iii) Find the critical path and the project duration.

Solution:





Activity	Duration	EST	LFT	LST	EFT	TF
1-2	2	0	7	5	2	5
1-3	2	0	2	0	2	0
1-4	1	0	7	6	1	6
2-5	4	2	11	7	6	5
3-6	8	2	10	2	10	0
3-7	5	2	8	3	7	1
4-6	3	1	10	7	4	6
5-8	1	6	12	11	7	5
6-9	5	10	15	10	15	0
7-8	4	7	12	8	11	1
8-9	3	11	15	12	14	1

The critical path is 1-3-6-9

The project duration is 15 months.

Problem 3:

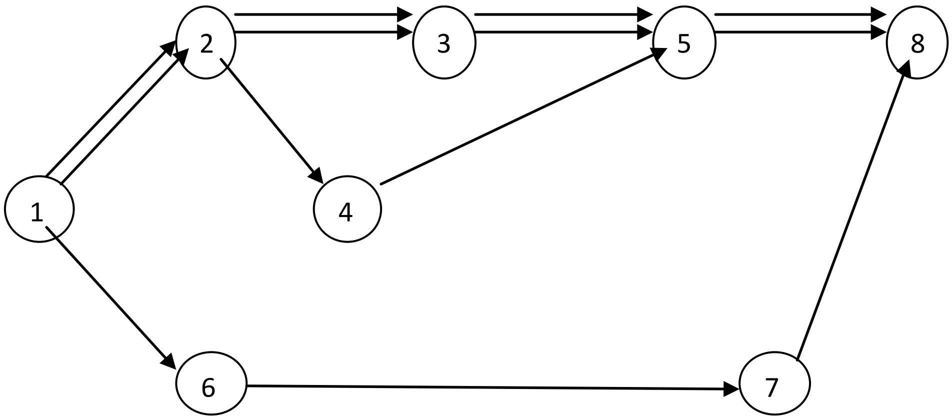
The activities of a project have the following PERT time estimates.

Job	Optimistic time	Most likely time	Pessimistic time
1-2	3	6	15
7-8	4	19	28
2-3	6	12	30
3-5	5	11	17
5-8	1	4	7
6-7	3	9	27
4-5	3	6	15
2-4	2	5	8
1-6	2	5	14

(i) Draw the network diagram and determine the critical path.

(ii) Find the project completion time and its variance.

Solution:



Job	$T_E = \frac{T_D + T_p + 4Tm}{6}$	$SD(\frac{T_P - T_O}{6})$	Variance
1-2	42/6=7	2	
7-8	108/6=18	4	16
2-3	84/6=14	4	16
3-5	66/6=11	2	4
5-8	24/6=4	1	1
6-7	66/6=11	4	16
4-5	42/6=7	2	4
2-4	30/6=5	1	1
1-6	36/6=6	2	4

The various paths are

		Duration
1.	1-2-3-5-8	36
2.	1-2-4-5-8	23
3.	1-6-7-8	35

The critical path is 1-2-3-5-8

The project duration is 36 days.

Variance of the critical path=4+16+4+1=25.

Problem 4:

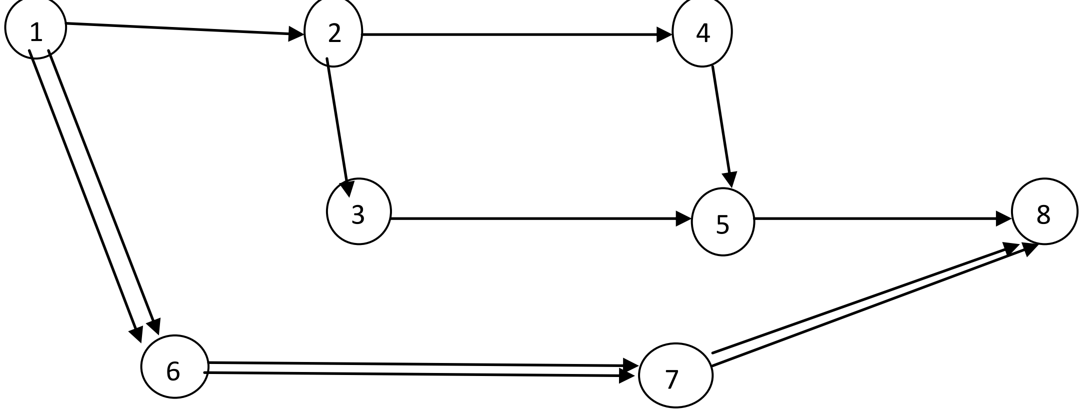
The following table lists the jobs of a network with their time estimates

Job	Optimistic	Duration days Most Likely	Pessimistic
1-2	3	6	15
1-6	2	5	14
2-3	6	12	30
2-4	2	5	8
3-5	5	11	17
4-5	3	6	15
6-7	3	9	27
5-8	1	4	7
7-8	4	19	28

Draw the project network and calculate the length and variance of the critical path.

Solution:

Job	$T_E = \frac{T_D + T_p + 4Tm}{6}$	$SD(\frac{T_P - T_O}{6})$	Variance
1-2	42/6=7	12/6=2	4
1-6	36/6=6	12/6=2	4
2-3	84/6=12	24/6=4	16
2-4	30/6=5	6/6=1	1
3-5	66/6=11	12/6=2	4
4-5	42/6=7	12/6=2	4
6-7	66/6=11	24/6=4	16
5-8	24/6=4	6/6=1	1
7-8	108/6=18	24/6=4	16



The various paths are

		Duration
1.	1-2-4-5-8	23
2.	1-2-3-5-8	36
3.	1-6-7-8	36

The critical path is 1-2-3-5-8 and 1-6-7-8.

The project duration is 36 days.

Variance of the critical path=4+16+16=36.

Exercises:

1. A project consists of 12 jobs. Draw a project network and determine the critical path.

Job	Duration	Job	Duration
1-2	2	6-7	8
2-3	7	6-10	4
2-4	3	7-9	4
3-4	3	8-9	1
3-5	5	9-10	7
4-6	3		
5-8	5		

2. Consider the following project whose activities along with PERT time estimates the optimistic time (a), most likely time (m), and the pessimistic time (b) are given as follows.

Activity	a (days)	m (days)	b (days)
1-2	12	14	21
1-3	7	10	16
3-5	4	6	10
3-4	36	40	60
4-6	12	15	24
5-6	6	8	12
6-7	9	12	18

6-8	6	10	15
7-8	4	5	7
8-9	8	10	14

****